Nuh Cards

EMh
-12

NACA TN No. 1832

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1832

SMALL BENDING AND STRETCHING OF SANDWICH-TYPE SHELLS

By Eric Reissner

Massachusetts Institute of Technology

Washington
March 1949

TABLE OF CONTENTS

Page

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1832

SMALL BENDING AND STRETCHING OF SANDWICH-TYPE SHELLS

By Eric Reissner

SUMMARY

A theory has been developed for small bending and stretching of sandwich-type shells. This theory is an extension of the known theory of homogeneous thin elastic shells. It was found that two effects are important in the present problem, which have not been considered previously in the theory of curved shells: (1) The effect of transverse shear deformation and (2) the effect of transverse normal stress deformation. The first of these two effects has been known to be of importance in the theory of plates and beams. The second effect was found to occur in a manner which is typical for shells and has no counterpart in flat-plate theory.

The general results of this report have been applied to the solution of problems concerning flat plates, circular rings, circular cylindrical shells, and spherical shells. In each case numerical examples have been given, illustrating the magnitude of the effects of transverse shear and normal stress deformation.

The results of this investigation indicate the necessity of taking account of transverse shear and normal stress in sandwich-type shells, as soon as there is an order-of-magnitude difference between the elastic constants of the core layer and of the face layers of the composite shell. It was found that the changes due to transverse shear and normal stress deformation in the core may be so large as to be no mere corrections to the results of the theory without transverse core flexibility.

The actual magnitude of the changes is greatly dependent on the geometry and loading condition of the structure under consideration so that no general rules may be given which indicate for which elastic modulus ratio the changes begin to be significant.

Solutions of problems in the present theory may in general be obtained by mathematical methods which are similar to those employed in the theory of plates and shells without the effect of transverse shear and normal stress deformation included. The present work does not include consideration of buckling and finite deflection effects.

INTRODUCTION

In this report an extension of the classical theory of small bending and stretching of thin elastic shells is considered. Instead of a homogeneous shell, investigation is made of a shell constructed in three layers: A core layer of thickness h with elastic constants E_c, G_c, and ν_c and two face layers of thickness t with elastic constants E_f, G_f, and ν_f. In the developments certain restrictive assumptions are made which somewhat limit the general applicability of the results. In so doing formulas are obtained which are as compact as possible while still describing the essential characteristics of the sandwich-type shell.

The thickness ratio t/h is assumed small compared with unity; at the same time the ratio $E_f t/E_c h$ is assumed large compared with unity. This latter assumption means that the face material is so much stiffer than the core material that the contribution of the core layer to stress couples and tangential stress resultants of the composite shell is negligible. It is known that for flat plates these assumptions necessitate the taking into account of the effect of transverse shear deformation. (See, for instance, reference 1.) The same would be expected to be true for curved shells, and the present report, therefore, gives a system of equations in which this effect is incorporated.

A further effect which, it appears, has not been considered previously in the analysis of small deflections of sandwich structures is the effect of transverse normal stress deformation. In the present report it is shown that this effect arises in a manner which is typical for shells and has no counterpart in plate theory. It may be likened, roughly, to what happens in the bending of curved tubes.

The process by which the general results of this report are obtained is as follows: First, each of the face layers of thickness t is assumed to behave like a thin shell without bending stiffness. The loads applied to these face shells, henceforth called face membranes, are of two kinds: (1) External loads and (2) loads caused by the stresses in the core layer. Next, the core layer of thickness h is assumed to behave like a three-dimensional elastic continuum in which those stresses which are parallel to the faces are negligible compared with the transverse shear and normal stresses. On the basis of these two assumptions three steps are carried out. First, the equilibrium equations of the core layer and of the face layers are obtained. Then an appropriate expression for the strain energy of the composite structure is derived. Finally, Castigliano's theorem of minimum complementary energy is used to obtain the relations which connect stress resultants and couples of the composite shell with the quantities which describe the state of deformation of the composite shell.

The system of equations which is obtained in the foregoing manner is specialized for the following cases:

(1) Flat plate

(2) Circular ring

(3) Circular cylindrical shell

(4) Spherical shell with axisymmetrical deformation

In each case a number of problems are solved explicitly and the appreciable effect of transverse shear and/or normal stress deformation is illustrated numerically.

This work was conducted at the Massachusetts Institute of Technology under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

SYMBOLS

h	core-layer thickness
t	face-layer thickness
ξ_1, ξ_2	curvilinear coordinates on middle surface of composite shell
ζ	distance coordinate measured along normal to middle surface of shell
α_1, α_2	coefficients of linear element on middle surface of shell
R_1, R_2	principal radii of curvature of middle surface of shell
N_{nmu}	direct stress resultants in upper face membrane; $n = 1,2$; $m = 1,2$
N_{nml}	direct stress resultants in lower face membrane
p_{nu}, p_{nl}	tangential components of external load intensity on upper and lower membranes
q_u, q_l	normal components of external load intensity on upper and lower membranes

$\tau_{1\zeta}$, $\tau_{2\zeta}$ — components of transverse shear stress in core layer

σ_ζ — component of transverse normal stress in core layer

$\tau_{n\zeta u}$, $\tau_{n\zeta l}$ — values of transverse shear stresses for $\zeta = \pm h/2$; $n = 1,2$

$\sigma_{\zeta u}$, $\sigma_{\zeta l}$ — values of transverse normal stresses for $\zeta = \pm h/2$

$\tau_{n\zeta m}$ — values of transverse shear stresses at middle surface of shell

Q_1, Q_2 — transverse shear stress resultants

N_{nm} — direct stress resultants parallel to middle surface for composite shell; $n = 1,2$; $m = 1,2$

M_{nm} — stress couples for composite shell; $n = 1,2$; $m = 1,2$

p_n — tangential components of external load intensity for composite shell; $n = 1,2$

q — normal component of external load intensity for composite shell

s — external load intensity term defined by equation (22)

π — strain energy

E_f, G_f, ν — elastic moduli of isotropic face-layer material; $\nu = \nu_f$

E_c, G_c — elastic moduli in transverse direction of core-layer material

u_1, u_2 — effective tangential components of displacement of elements of composite shell

w — effective normal component of displacement of elements of composite shell

β_1, β_2 — effective components of change of slope of normal to middle surface of composite shell

$\epsilon_{\zeta m}$ — component of strain $\left(\epsilon_{\zeta m} = \sigma_{\zeta m}/E_c\right)$

$C^* = 2tE_f$

$D^* = (1/2)t(h + t)^2 E_f$

$C = C^*/(1 - \nu^2)$

D	bending stiffness factor $\left(D = D^*/(1 - \nu^2)\right)$
x, y	Cartesian coordinates in plane of flat plate
r, θ	polar coordinates in plane of flat plate
a	radius of circular ring, cylindrical shell, and spherical shell
x, θ	surface coordinates on cylindrical shell
$\lambda_1, \lambda_2, \lambda_{12}$	parameters defined by equation (63)
$\mu = \pi/l$	
l	half wave length of sinusoidal load distribution
m_1, m_2	quantities defined by equation (197)
k	complex quantity defined by equation (200)
ϕ, θ	surface coordinates on spherical shell
ω	quantity defined by equation (74)
K	parameter defined by equation (190)

I – GENERAL THEORY

Statics of Sandwich-Type Shell

In order to derive a complete system of equations for the shell composed of face layers and core layers it is necessary first to consider separately the statics of the face layers and that of the core layer of the shell. Combination of the results obtained for these two components of the composite structure must and will lead to those differential equations of equilibrium which hold for elements of a shell, whether this shell is of homogeneous or nonhomogeneous construction. In addition, however, relations are obtained which are characteristic of the sandwich-type shell.

Coordinate system on shell.– A curvilinear coordinate system (ξ_1, ξ_2, ζ) is introduced as follows: Let ξ_1 and ξ_2 be coordinates

on the middle surface of the composite shell and let ζ be the distance of a point of the shell from its middle surface, measured along the normal to the middle surface. In order that this system of coordinates be an orthogonal system, choose the ξ_1, ξ_2 curves as lines of curvature on the middle surface (in the case of shells of revolution the lines of curvature are identical with the meridians and parallels on the middle surface).

The linear element in the foregoing system of coordinates is of the form

$$ds^2 = \alpha_1{}^2\left(1 + \frac{\zeta}{R_1}\right)^2 d\xi_1{}^2 + \alpha_2{}^2\left(1 + \frac{\zeta}{R_2}\right)^2 d\xi_2{}^2 + d\zeta^2 \tag{1}$$

where α_1 and α_2 are the coefficients of the linear element on the middle surface and R_1 and R_2 are the principal radii of curvature of the middle surface (see fig. 1). Formulas for the calculation of the quantities α_n and R_n are contained in texts on differential geometry. They are collected, together with other results, in reference 2, which deals with the theory of homogeneous thin shells.

Statics of face layers.- The face layers are treated as thin shells of thickness t and it is assumed that the bending stiffness of these thin shells about their own middle surface may be neglected.[1] Because of this neglect from now on they will be designated as face membranes.

The middle surfaces of the face membranes evidently are given with reference to the three-dimensional system of curvilinear coordinates by $\zeta = \frac{1}{2}(h + t)$ and $\zeta = -\frac{1}{2}(h + t)$. From equation (1) it follows that the linear element on the middle surfaces of the face membranes is given by

$$ds^2 = \alpha_1{}^2\left(1 \pm \frac{h + t}{2R_1}\right)^2 d\xi_1{}^2 + \alpha_2{}^2\left(1 \pm \frac{h + t}{2R_2}\right) d\xi_2{}^2 + d\zeta^2 \tag{2}$$

The components of external load intensity on the upper and lower membranes are designated by p_{1u}, p_{2u}, and q_u and by p_{1l}, p_{2l}, and q_l, respectively (fig. 2). The core-layer stresses which act on the upper and lower membranes are given as $\tau_{1\zeta u}$, $\tau_{2\zeta u}$, and $\sigma_{\zeta u}$ and by $\tau_{1\zeta l}$, $\tau_{2\zeta l}$,

[1] This, of course, means that no local buckling phenomena are considered in the present work.

and $\sigma_{\zeta l}$, respectively. Finally, the direct stress resultants in the upper and lower face membranes are designated by N_{11u}, N_{12u}, N_{21u}, and N_{22u} and by N_{11l}, N_{12l}, N_{21l}, and N_{22l}, respectively (fig. 2).

There are then three equations of force equilibrium for the elements of each of the two membranes. Writing

$$\left.\begin{aligned} \alpha_{nu} &= \alpha_n\left(1 + \frac{h + t}{2R_n}\right) \\ \alpha_{nl} &= \alpha_n\left(1 - \frac{h + t}{2R_n}\right) \end{aligned}\right\} \tag{3}$$

the equations for the upper-face membrane are the following[2]:

$$\frac{\partial \alpha_{2u} N_{11u}}{\partial \xi_1} + \frac{\partial \alpha_{1u} N_{21u}}{\partial \xi_2} + N_{12u}\frac{\partial \alpha_{1u}}{\partial \xi_2} - N_{22u}\frac{\partial \alpha_{2u}}{\partial \xi_1} + \alpha_{1u}\alpha_{2u}\left(p_{1u} - \tau_{1\zeta u}\right) = 0 \tag{4}$$

$$\frac{\partial \alpha_{2u} N_{12u}}{\partial \xi_1} + \frac{\partial \alpha_{1u} N_{22u}}{\partial \xi_2} + N_{21u}\frac{\partial \alpha_{2u}}{\partial \xi_1} - N_{11u}\frac{\partial \alpha_{1u}}{\partial \xi_2} + \alpha_{1u}\alpha_{2u}\left(p_{2u} - \tau_{2\zeta u}\right) = 0 \tag{5}$$

$$\alpha_{1u}\alpha_{2u}\left[\frac{N_{11u}}{R_1\left(1 + \frac{h + t}{2R_1}\right)} + \frac{N_{22u}}{R_2\left(1 + \frac{h + t}{2R_2}\right)} - q_u + \sigma_{\zeta u}\right] = 0 \tag{6}$$

The corresponding equations for the lower-face membrane are

$$\frac{\partial \alpha_{2l} N_{11l}}{\partial \xi_1} + \frac{\partial \alpha_{1l} N_{21l}}{\partial \xi_2} + N_{12l}\frac{\partial \alpha_{1l}}{\partial \xi_2} - N_{22l}\frac{\partial \alpha_{2l}}{\partial \xi_1} + \alpha_{1l}\alpha_{2l}\left(p_{1l} + \tau_{1\zeta l}\right) = 0 \tag{7}$$

$$\frac{\partial \alpha_{2l} N_{12l}}{\partial \xi_1} + \frac{\partial \alpha_{1l} N_{22l}}{\partial \xi_2} + N_{21l}\frac{\partial \alpha_{2l}}{\partial \xi_1} - N_{11l}\frac{\partial \alpha_{1l}}{\partial \xi_2} + \alpha_{1l}\alpha_{2l}\left(p_{2l} + \tau_{2\zeta l}\right) = 0 \tag{8}$$

---

[2]These are obtained from the corresponding equations of reference 2 with α_n changed to α_{nu} and with stress couples and transverse shear stress resultants omitted. To make up for this omission, the loads on the two membranes are assumed to act at their middle surfaces; this means terms of the order t/R are neglected (but not terms of order h/R).

$$\alpha_{1l}\alpha_{2l}\left[\frac{N_{11l}}{R_1\left(1 - \frac{h + t}{2R_1}\right)} + \frac{N_{22l}}{R_2\left(1 - \frac{h + t}{2R_2}\right)} - q_l - \sigma\zeta_l\right] = 0 \tag{9}$$

As bending moments and transverse shears are assumed not to be acting in the individual membranes the moment equilibrium equations become the symmetry relations

$$\left.\begin{aligned} N_{12u} &= N_{21u} \\ N_{12l} &= N_{21l} \end{aligned}\right\} \tag{10}$$

Before analyzing the state of stress in the core layer it is convenient to see what relations follow from equations (4) to (9) for the composite shell.

Statics of composite shell.— It may be seen that, in view of the fact that all face-parallel stresses in the core layer are neglected, the following expressions for the face-parallel stress resultants and couples of the composite shell are obtained:

$$N_{11} = \left(1 + \frac{h + t}{2R_2}\right)N_{11u} + \left(1 - \frac{h + t}{2R_2}\right)N_{11l} \tag{11}$$

$$N_{12} = \left(1 + \frac{h + t}{2R_2}\right)N_{12u} + \left(1 - \frac{h + t}{2R_2}\right)N_{12l} \tag{12}$$

$$N_{21} = \left(1 + \frac{h + t}{2R_1}\right)N_{12u} + \left(1 - \frac{h + t}{2R_1}\right)N_{12l} \tag{13}$$

$$N_{22} = \left(1 + \frac{h + t}{2R_1}\right)N_{22u} + \left(1 - \frac{h + t}{2R_1}\right)N_{22l} \tag{14}$$

$$M_{11} = \frac{h + t}{2}\left[\left(1 + \frac{h + t}{2R_2}\right)N_{11u} - \left(1 - \frac{h + t}{2R_2}\right)N_{11l}\right] \tag{15}$$

$$M_{12} = \frac{h + t}{2}\left[\left(1 + \frac{h + t}{2R_2}\right)N_{12u} - \left(1 - \frac{h + t}{2R_2}\right)N_{12l}\right] \tag{16}$$

$$M_{21} = \frac{h + t}{2}\left[\left(1 + \frac{h + t}{2R_1}\right)N_{12u} - \left(1 - \frac{h + t}{2R_1}\right)N_{12l}\right] \qquad (17)$$

$$M_{22} = \frac{h + t}{2}\left[\left(1 + \frac{h + t}{2R_1}\right)N_{22u} - \left(1 - \frac{h + t}{2R_1}\right)N_{22l}\right] \qquad (18)$$

In the same way the following expressions are obtained for components of external force and moment intensity:

$$p_n = \left(1 + \frac{h + t}{2R_2}\right)\left(1 + \frac{h + t}{2R_1}\right)p_{nu} + \left(1 - \frac{h + t}{2R_2}\right)\left(1 - \frac{h + t}{2R_1}\right)p_{nl} \qquad (19)$$

$$q = \left(1 + \frac{h + t}{2R_2}\right)\left(1 + \frac{h + t}{2R_1}\right)q_u + \left(1 - \frac{h + t}{2R_2}\right)\left(1 - \frac{h + t}{2R_1}\right)q_l \qquad (20)$$

$$m_n = \frac{h + t}{2}\left[\left(1 + \frac{h + t}{2R_2}\right)\left(1 + \frac{h + t}{2R_1}\right)p_{nu} - \left(1 - \frac{h + t}{2R_2}\right)\left(1 - \frac{h + t}{2R_1}\right)p_{nl}\right] \qquad (21)$$

Further, a load term of the following form will be encountered:

$$s = \frac{1}{2}\left[\left(1 + \frac{h + t}{2R_2}\right)\left(1 + \frac{h + t}{2R_1}\right)q_u - \left(1 - \frac{h + t}{2R_2}\right)\left(1 - \frac{h + t}{2R_1}\right)q_l\right] \qquad (22)$$

which bears a relation to equation (20) similar to that which equation (21) bears to equation (19). This last term would represent, for a homogeneous shell, the average transverse normal stress at any station of the shell, assuming that the loads q_u and q_l alone are responsible for this stress. For a homogeneous isotropic shell this term is of no importance. For a sandwich-type shell, as will be seen, it may sometimes be of importance.

In order to obtain force and moment equilibrium equations for the composite shell the face-membrane equilibrium equations (4) to (9) are combined suitably. Adding equations (4) and (7), and (5) and (8), respectively, the two equilibrium equations for the force components parallel to the middle surface of the shell are obtained. In order to reduce them to known form (see reference 2) the following relations are used between the core-layer-surface shear stresses $\tau_{n\zeta u}$ and $\tau_{n\zeta l}$ and the transverse shear stress resultants Q_1 and Q_2.

$$\left(1 + \frac{h + t}{2R_1}\right)\left(1 + \frac{h + t}{2R_2}\right)\tau_{n\zeta u} - \left(1 - \frac{h + t}{2R_1}\right)\left(1 - \frac{h + t}{2R_2}\right)\tau_{n\zeta l} = -\frac{Q_n}{R_n} \tag{23}$$

$$\frac{h + t}{2}\left[\left(1 + \frac{h + t}{2R_1}\right)\left(1 + \frac{h + t}{2R_2}\right)\tau_{n\zeta u} + \left(1 - \frac{h + t}{2R_1}\right)\left(1 - \frac{h + t}{2R_2}\right)\tau_{n\zeta l}\right] = Q_n \tag{24}$$

Equations (23) and (24) will subsequently be shown to be in agreement with the usual definition for the transverse shear stress resultants by consideration of the stress distribution of the core layer.

With equations (23) and (24), there are obtained by combination of equations (4) and (7), and (5) and (8) – carrying out addition as well as subtraction – the following four equations:

$$\frac{\partial\alpha_2 N_{11}}{\partial\xi_1} + \frac{\partial\alpha_1 N_{21}}{\partial\xi_2} + N_{12}\frac{\partial\alpha_1}{\partial\xi_2} - N_{22}\frac{\partial\alpha_2}{\partial\xi_1} + \alpha_1\alpha_2\left(\frac{Q_1}{R_1} + p_1\right) = 0 \tag{25)*$$

$$\frac{\partial\alpha_2 N_{12}}{\partial\xi_1} + \frac{\partial\alpha_1 N_{22}}{\partial\xi_2} + N_{21}\frac{\partial\alpha_2}{\partial\xi_1} - N_{11}\frac{\partial\alpha_1}{\partial\xi_2} + \alpha_1\alpha_2\left(\frac{Q_2}{R_2} + p_2\right) = 0 \tag{26)*$$

$$\frac{\partial\alpha_2 M_{11}}{\partial\xi_1} + \frac{\partial\alpha_1 M_{21}}{\partial\xi_2} + M_{12}\frac{\partial\alpha_1}{\partial\xi_2} - M_{22}\frac{\partial\alpha_2}{\partial\xi_1} + \alpha_1\alpha_2(m_1 - Q_1) = 0 \tag{27)*$$

$$\frac{\partial\alpha_2 M_{12}}{\partial\xi_1} + \frac{\partial\alpha_1 M_{22}}{\partial\xi_2} + M_{21}\frac{\partial\alpha_2}{\partial\xi_1} - M_{11}\frac{\partial\alpha_1}{\partial\xi_2} + \alpha_1\alpha_2(m_2 - Q_2) = 0 \tag{28)*$$

Two further equations are obtained by adding and subtracting, respectively, equations (6) and (9). Adding equations (6) and (9) and taking account of equations (11), (14), and (20), there follows:

$$\begin{aligned}\alpha_1\alpha_2\Bigg[\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}\right) - q + \left(1 + \frac{h + t}{2R_2}\right)\left(1 + \frac{h + t}{2R_1}\right)\sigma_{\zeta u} \\ - \left(1 - \frac{h + t}{2R_2}\right)\left(1 - \frac{h + t}{2R_1}\right)\sigma_{\zeta l}\Bigg] = 0\end{aligned} \tag{29}$$

In order that this reduces to the correct equation of transverse force equilibrium as given in reference 2,

$$-\alpha_1\alpha_2\left[\left(1+\frac{h+t}{2R_2}\right)\left(1+\frac{h+t}{2R_1}\right)\sigma_{\zeta u}-\left(1-\frac{h+t}{2R_2}\right)\left(1-\frac{h+t}{2R_1}\right)\sigma_{\zeta l}\right]$$
$$=\frac{\partial\alpha_2 Q_1}{\partial\xi_1}+\frac{\partial\alpha_1 Q_2}{\partial\xi_2} \tag{30}$$

Equation (30), just as equations (23) and (24), can again be verified independently by consideration of the state of stress in the core layer. On the basis of equation (30), equation (29) is written in the form

$$\frac{\partial\alpha_2 Q_1}{\partial\xi_1}+\frac{\partial\alpha_1 Q_2}{\partial\xi_2}-\alpha_1\alpha_2\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)+\alpha_1\alpha_2 q=0 \tag{31)*$$

The last equation, use of which is required for the sandwich-type shell and which has not previously been given, is obtained by subtracting equation (9) from equation (6). Taking account of equations (15), (18), and (22), there results

$$\frac{2\alpha_1\alpha_2}{h+t}\left(\frac{M_{11}}{R_1}+\frac{M_{22}}{R_2}\right)-2\alpha_1\alpha_2 s+\alpha_1\alpha_2\left[\left(1+\frac{h+t}{2R_2}\right)\left(1+\frac{h+t}{2R_1}\right)\sigma_{\zeta u}\right.$$
$$\left.+\left(1-\frac{h+t}{2R_2}\right)\left(1-\frac{h+t}{2R_1}\right)\sigma_{\zeta l}\right]=0 \tag{32}$$

Provisionally, there is written

$$\left(1+\frac{h+t}{2R_2}\right)\left(1+\frac{h+t}{2R_1}\right)\sigma_{\zeta u}+\left(1-\frac{h+t}{2R_2}\right)\left(1-\frac{h+t}{2R_1}\right)\sigma_{\zeta l}=2\sigma_{\zeta m} \tag{33}$$

and it will subsequently be shown that σ_{ζ_m} represents the value of σ_ζ at the middle surface of the shell. Combining equations (33) and (32) yields

$$\sigma_{\zeta_m} + \frac{1}{h + t}\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right) - s = 0 \tag{34*}$$

Equation (34) has no relation to the sixth equation of equilibrium for an element of the shell which expresses the condition of moment equilibrium about the normal to the middle surface. That equation which, as is known, is an identity when resultants and couples are expressed in terms of stresses does not occur in the present derivations, or rather it is contained in equations (12), (13), (16), and (17), which give explicitly the slight differences between N_{12} and N_{21}, and M_{12} and M_{21}.

Stress distribution in core layer.- In order to verify independently equations (23), (24), and (30), as well as for the subsequent derivation of appropriate stress-strain relations, it is necessary to determine the distribution of stress in the core layer.

Assuming that the components of stress σ_1, σ_2, and τ_{12} in the core which would contribute to stress resultants and couples of the composite shell are of negligible importance,[3] these components of stress may be set equal to zero and only the components of transverse shear stress and transverse normal stress $\tau_{1\zeta}$, $\tau_{2\zeta}$, and σ_ζ may be retained. The differential equations of equilibrium for these three remaining components of stress in the system of curvilinear coordinates defined by equation (1) are obtained, from the general form of these differential equations in reference 3, in the following form:

$$\frac{\partial}{\partial\zeta}\left[\left(1 + \frac{\zeta}{R_1}\right)^2\left(1 + \frac{\zeta}{R_2}\right)\tau_{1\zeta}\right] = 0 \tag{35}$$

$$\frac{\partial}{\partial\zeta}\left[\left(1 + \frac{\zeta}{R_2}\right)^2\left(1 + \frac{\zeta}{R_1}\right)\tau_{2\zeta}\right] = 0 \tag{36}$$

[3]It is for this purpose that the order-of-magnitude relation $hE_c/tE_f \ll 1$ is assumed.

$$\frac{\partial}{\partial\zeta}\left[\alpha_1\alpha_2\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\sigma_\zeta\right] + \frac{\partial}{\partial\xi_1}\left[\alpha_2\left(1 + \frac{\zeta}{R_2}\right)\tau_{1\zeta}\right]$$

$$+ \frac{\partial}{\partial\xi_2}\left[\alpha_1\left(1 + \frac{\zeta}{R_1}\right)\tau_{2\zeta}\right] = 0 \tag{37}$$

The values of the three stress components at the middle surface ($\zeta = 0$) are designated by the subscript m. Integration of equations (35) to (37) then gives

$$\tau_{1\zeta} = \frac{\tau_{1\zeta_m}}{\left(1 + \zeta/R_1\right)^2\left(1 + \zeta/R_2\right)} \tag{38}$$

$$\tau_{2\zeta} = \frac{\tau_{2\zeta_m}}{\left(1 + \zeta/R_2\right)^2\left(1 + \zeta/R_1\right)} \tag{39}$$

$$\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\sigma_\zeta = \sigma_{\zeta_m} - \frac{\zeta}{\alpha_1\alpha_2}\left[\frac{\partial}{\partial\xi_1}\left(\frac{\alpha_2\tau_{1\zeta_m}}{1 + \zeta/R_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{\alpha_1\tau_{2\zeta_m}}{1 + \zeta/R_2}\right)\right] \tag{40}$$

The transverse shear stress resultants Q_1 and Q_2 are obtained from equations (38) and (39) in the form[4]

$$Q_n = \int_{-(h+t)/2}^{(h+t)/2} \tau_{n\zeta}\left(1 + \frac{\zeta}{R_m}\right)d\zeta = -R_n\tau_{n\zeta_m}\left(\frac{1}{1 + \frac{h + t}{2R_n}} - \frac{1}{1 - \frac{h + t}{2R_n}}\right)$$

$$= \frac{(h + t)\tau_{n\zeta_m}}{1 - \left(\frac{h + t}{2R_n}\right)^2} \tag{41}$$

[4]The integration must be extended over the thickness of the core layer and also over half the thickness of the face layers, in accordance with the prior assumption that the stresses $\tau_{n\zeta_u}$, $\tau_{n\zeta_l}$, σ_{ζ_u}, and σ_{ζ_l} may be taken to act at the middle surfaces of the respective face membranes.

Now, as intended, the proof is carried out of equations (23), (24), and (30), which were used to obtain the differential equations for the composite shell.

To verify equation (23), from equations (38) and (39) for the left-hand side of equation (23), the following equation is obtained:

$$\frac{\tau_{n\zeta m}}{1 + \frac{h + t}{2R_n}} - \frac{\tau_{n\zeta m}}{1 - \frac{h + t}{2R_n}} = -\frac{h + t}{R_n} \frac{\tau_{n\zeta m}}{1 - \left(\frac{h + t}{2R_n}\right)^2}$$

and this, in conjunction with equation (41), verifies equation (23).

To verify equation (24) in the same manner, from equations (38) and (39) for the left side of equation (24), the following equation is obtained:

$$\frac{h + t}{2}\left(\frac{\tau_{n\zeta m}}{1 + \frac{h + t}{2R_n}} + \frac{\tau_{n\zeta m}}{1 - \frac{h + t}{2R_n}}\right) = \frac{(h + t)\tau_{n\zeta m}}{1 - \left(\frac{h + t}{2R_n}\right)^2}$$

and this, in conjunction with equation (41), verifies equation (24).

To verify equation (30), equation (40) is used to write for the left side of equation (30)

$$\frac{\partial}{\partial \xi_1}\left(\frac{\alpha_2 \tau_{1\zeta m}}{1 + \frac{h + t}{2R_1}}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{\alpha_1 \tau_{2\zeta m}}{1 + \frac{h + t}{2R_2}}\right) + \frac{\partial}{\partial \xi_1}\left(\frac{\alpha_2 \tau_{1\zeta m}}{1 - \frac{h + t}{2R_1}}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{\alpha_1 \tau_{2\zeta m}}{1 - \frac{h + t}{2R_2}}\right)$$

$$= \frac{\partial}{\partial \xi_1}\left(\frac{2\alpha_2 \tau_{1\zeta m}}{1 - \left(\frac{h + t}{2R_1}\right)^2}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{2\alpha_1 \tau_{2\zeta m}}{1 - \left(\frac{h + t}{2R_2}\right)^2}\right)$$

and this, in conjunction with equation (41), verifies equation (30).

The section on the stress distribution in the core layer is concluded by listing the form which equations (38) to (40) for the stresses in the core layers assume for "thin" shells, that is, for shells for which $h/R \ll 1$. From equations (38) and (39), in conjunction with equation (41), it follows that

$$\left.\begin{aligned} \tau_{1\zeta} &= \frac{Q_1}{h + t} \\ \tau_{2\zeta} &= \frac{Q_2}{h + t} \end{aligned}\right\} \qquad (42)*$$

From equation (40), in conjunction with equation (41), it follows that

$$\alpha_1\alpha_2\sigma_\zeta = \alpha_1\alpha_2\sigma_{\zeta m} - \frac{\zeta}{h + t}\left(\frac{\partial\alpha_2 Q_1}{\partial\xi_1} + \frac{\partial\alpha_1 Q_2}{\partial\xi_2}\right) \qquad (43a)$$

It is necessary to note for some of the following considerations that, in view of equation (31), instead of equation (43a) there may be written

$$\sigma_\zeta = \sigma_{\zeta m} - \frac{\zeta}{h + t}\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - q\right) \qquad (43b)*$$

It is seen that in this approximation the transverse shear stresses are uniform across the thickness of the core layer, while the transverse normal stress is composed of two terms, one uniform across the thickness and the other varying linearly across the thickness.

No further calculations are needed with reference to the state of stress in the composite shell. The next step is to complete the system of differential equations for stress resultants and couples by deriving an appropriate system of stress-strain relations.

Strain Energy of Sandwich-Type Shell

In calculating the strain energy of face membranes and core layer it is assumed that both are isotropic and elastic, with elastic constants E_f, $\nu_f = \nu$, $G_f = E_f/2(1 + \nu)$ and E_c, ν_c, $G_c = E_c/2(1 + \nu_c)$.

Poisson's ratio for the face membranes is written without a subscript, because, in view of the assumed stress distribution, there is no explicit occurrence of Poisson's ratio ν_c for the core layer.

The strain energy for the composite shell is the sum of the strain energies for the face membranes and for the core layer

$$\pi = \pi_f + \pi_c \tag{44}$$

For the purpose of obtaining stress-strain relations, both π_f and π_c are expressed in terms of stresses rather than in terms of strains.

<u>Strain energy of face layers</u>.- Considering that the element of area on the middle surfaces of the membranes is of the form $\alpha_1\alpha_2\left(1 \pm \frac{h + t}{2R_1}\right)\left(1 \pm \frac{h + t}{2R_2}\right) d\xi_1 \, d\xi_2$ and that the stresses in the membranes are the stress resultants divided by the membrane thickness t, there is, from well-known principles, the following relation:

$$\begin{aligned}
\pi_f = {} & \frac{1}{2}\iint \frac{1}{tE_f}\left[N_{11u}^2 + N_{22u}^2 - 2\nu N_{11u}N_{22u} + 2(1 + \nu)N_{12u}^2\right] \\
& \times \left(1 + \frac{h + t}{2R_1}\right)\left(1 + \frac{h + t}{2R_2}\right)\alpha_1\alpha_2 \, d\xi_1 \, d\xi_2 \\
& + \frac{1}{2}\iint \frac{1}{tE_f}\left[N_{11l}^2 + N_{22l}^2 - 2\nu N_{11l}N_{22l} + 2(1 + \nu)N_{12l}^2\right] \\
& \times \left(1 - \frac{h + t}{2R_1}\right)\left(1 - \frac{h + t}{2R_2}\right)\alpha_1\alpha_2 \, d\xi_1 \, d\xi_2
\end{aligned} \tag{45}$$

Equation (45) is transformed into an expression containing stress resultants and couples of the composite shell by means of equations (11) to (18) which lead to the relations

$$\left.\begin{aligned} 2\left(1 + \frac{h + t}{2R_2}\right)N_{11u} &= N_{11} + \frac{2}{h + t} M_{11} \\ 2\left(1 - \frac{h + t}{2R_2}\right)N_{11l} &= N_{11} - \frac{2}{h + t} M_{11} \end{aligned}\right\} \tag{46}$$

with corresponding formulas for N_{12} and N_{22}.[5]

In this transformation the cases are limited to those for which $h/R \ll 1$. Then, with the two constants C^* and D^* defined by

$$\left.\begin{aligned} C^* &= 2tE_f \\ D^* &= \frac{1}{2} t(h + t)^2 E_f \end{aligned}\right\} \tag{47}$$

the following expression for π_f is obtained:

$$\pi_f = \frac{1}{2} \iint \left\{ \frac{1}{C^*}\left[N_{11}^2 + N_{22}^2 - 2\nu N_{11}N_{22} + 2(1 + \nu)N_{12}^2\right] + \frac{1}{D^*}\left[M_{11}^2 + M_{22}^2 - 2\nu M_{11}M_{22} + 2(1 + \nu)M_{12}^2\right]\right\} \alpha_1\alpha_2 \, d\xi_1 \, d\xi_2 \qquad (48)^*$$

It may be remarked that equation (48) could have been given directly, by analogy with known results for the isotropic homogeneous shell.

Strain energy of core layer.— With the stresses σ_1, σ_2, and τ_{12} assumed to vanish, there results for the strain energy of the core layer

---

[5]Note that equations (46) and corresponding equations can be used to calculate the stresses in the two different face membranes, once stress resultants and couples in the composite shell are known.

$$\pi_c = \frac{1}{2}\iiint_{-\frac{h+t}{2}}^{\frac{h+t}{2}}\left(\frac{\sigma_\zeta{}^2}{E_c} + \frac{\tau_{1\zeta}{}^2 + \tau_{2\zeta}{}^2}{G_c}\right)\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) d\zeta\alpha_1\alpha_2\ d\xi_1\ d\xi_2 \tag{49}$$

Again the terms ζ/R compared with unity are neglected and, consistent with this neglect, the values of the stresses $\tau_{n\zeta}$ and σ_ζ are taken from equations (42) and (43).

The value of σ_ζ may be chosen from either equation (43a) or equation (43b). The form of the results depends somewhat on which of the two equations is chosen, in the sense that the meaning of the deformation quantities which are to be determined depends on which of the two equations is taken. This question is decided in the following manner: As all resultants and couples enter the expression for the strain energy only as themselves and not in differentiated form, except when equation (43a) is used, the selection of equation (43b) for σ_ζ is proposed, thereby excluding derivatives of stress resultants and couples from the expression for the strain energy π.

Introducing then equation (43b) into equation (49) yields

$$\pi_c = \frac{1}{2}\iiint\left\{\frac{Q_1{}^2 + Q_2{}^2}{(h + t)^2G_c} + \frac{1}{E_c}\left[\sigma_{\zeta m} - \frac{\zeta}{h + t}\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - q\right)\right]^2\right\}$$

$$\times\ d\zeta\alpha_1\alpha_2\ d\xi_1\ d\xi_2 \tag{50}$$

The integration with respect to ζ is carried out and equation (50) becomes

$$\pi_c = \frac{1}{2}\iint\left\{\frac{Q_1{}^2 + Q_2{}^2}{(h + t)G_c} + \frac{h + t}{E_c}\left[\sigma_{\zeta m}{}^2 + \frac{1}{12}\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - q\right)^2\right]\right\}\alpha_1\alpha_2\ d\xi_1\ d\xi_2 \tag{51*}$$

It was to be expected that the terms containing the modulus of rigidity G_c would occur in the foregoing form. The contribution of

the present report up to this point, besides giving the new equation (34) for $\sigma_{\zeta m}$, is thought to be the determination of the form in which the effect of transverse normal stress deformability manifests itself in the strain energy of the sandwich shell.

Stress-Strain Relations for Composite Shell

In what follows a system of stress-strain relations for the composite shell is obtained by the use of Castigliano's theorem of minimum complementary energy. The manner in which the theorem is used here appears to have been employed first by E. Trefftz (reference 4) for the purpose of avoiding geometrical considerations in the derivation of the stress-strain relations for thin homogeneous shells with small deformations, without consideration of the effects of transverse shear and normal stress deformation.

Assuming for the present purpose that all boundary conditions for the shell under consideration are stress conditions, the theorem consists in the statement that among all statically correct states of stress the actually occurring state of stress makes the strain energy of the system a minimum. In the application of the theorem the fact is taken into account that statically correct states of stress only are to be compared, by means of the Lagrangian multiplier method. Before minimizing π an integral is added to it which contains the six equilibrium equations (25) to (28), (31), and (34), each of the six equations multiplied by a Lagrangian multiplier. It can then be shown, by using Castigliano's theorem with prescribed boundary displacements instead of with prescribed boundary stresses, that each of the six multipliers has the meaning of one of the displacement quantities which occur in the shell problem.[6]

With the foregoing understanding of the meaning of the multipliers, the multiplier of equation (25) is designated by u_1; that of equation (26), by u_2; that of equation (27), by β_1; that of equation (28), by β_2; that of equation (31), by w; and finally that of equation (34), by k. It is known that u_1, u_2, and w represent the effective components of displacement in the ξ_1, ξ_2, and ζ directions, respectively. Further, it is known that β_1 and β_2 represent the angles through which the normal to the middle surface of the shell turns toward the ξ_1 and ξ_2 curves, respectively. There is no

[6]For the special case of the flat plate this has been carried out explicitly in reference 1. For the case of the homogeneous shell, without effect of transverse shear and normal stress deformation, the proof has been given in reference 4. The proof for the more general case which is here considered is not included as it does not offer any clearer insight into the problem and tends to lengthen the analytical discussion.

immediate simple geometrical interpretation for k and, while such interpretation in terms of an average transverse normal strain might be deduced herein, k is considered as an auxiliary variable presently to be eliminated.

Combining now equations (44), (48), (51), and (25) to (28), (31), and (34) in the manner indicated, the following variational equation results:

$$
\begin{aligned}
\frac{1}{2}\,\delta \iint \Bigg\{ & \frac{1}{C^*}\left[N_{11}{}^2 + N_{22}{}^2 - 2\nu N_{11}N_{22} + 2(1+\nu)N_{12}{}^2\right] \\
& + \frac{1}{D^*}\left[M_{11}{}^2 + M_{22}{}^2 - 2\nu M_{11}M_{22} + 2(1+\nu)M_{12}{}^2\right] \\
& + \frac{Q_1{}^2 + Q_2{}^2}{(h+t)G_c} + \frac{h+t}{E_c}\left[\sigma\zeta_m{}^2 + \frac{1}{12}\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - q\right)^2\right]\Bigg\}\alpha_1\alpha_2\, d\xi_1\, d\xi_2 \\
& + \delta \iint \Bigg\{ u_1\left[\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{21}}{\partial \xi_2} + N_{12}\frac{\partial \alpha_1}{\partial \xi_2} - N_{22}\frac{\partial \alpha_2}{\partial \xi_1}\right. \\
& \left. + \alpha_1\alpha_2\left(\frac{Q_1}{R_1} + p_1\right)\right] + u_2\left[\frac{\partial \alpha_2 N_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2} + N_{21}\frac{\partial \alpha_2}{\partial \xi_1} - N_{11}\frac{\partial \alpha_1}{\partial \xi_2}\right. \\
& \left. + \alpha_1\alpha_2\left(\frac{Q_2}{R_2} + p_2\right)\right] + \beta_1\left[\frac{\partial \alpha_2 M_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{21}}{\partial \xi_2} + M_{12}\frac{\partial \alpha_1}{\partial \xi_2} - M_{22}\frac{\partial \alpha_2}{\partial \xi_1}\right. \\
& \left. + \alpha_1\alpha_2(m_1 - Q_1)\right] + \beta_2\left[\frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{22}}{\partial \xi_2} + M_{21}\frac{\partial \alpha_2}{\partial \xi_1} - M_{11}\frac{\partial \alpha_1}{\partial \xi_2}\right. \\
& \left. + \alpha_1\alpha_2(m_2 - Q_2)\right] + w\left[\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2} - \alpha_1\alpha_2\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - q\right)\right] \\
& + k\left[\sigma\zeta_m + \frac{1}{h+t}\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right) - s\right]\Bigg\}\, d\xi_1\, d\xi_2 = 0
\end{aligned}
\tag{52}
$$

The variations in equation (52) are carried out and integration is done by parts to eliminate derivatives of variations in the double integral. The line integrals along the boundary which occur due to this integration by parts vanish, because it has been assumed that all stresses are prescribed at the boundary and therefore their variations vanish at the boundary.

The resultant variational equation is

$$\iint \Bigg\{ \delta N_{11}\left[\frac{N_{11} - \nu N_{22}}{C^*} - \frac{1}{\alpha_1}\frac{\partial u_1}{\partial \xi_1} - \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} - \frac{w}{R_1} + \frac{h + t}{12E_cR_1}\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - q\right)\right]$$

$$+ \delta N_{22}\left[\frac{N_{22} - \nu N_{11}}{C^*} - \frac{1}{\alpha_2}\frac{\partial u_2}{\partial \xi_2} - \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} - \frac{w}{R_2}\right.$$

$$\left. + \frac{h + t}{12E_cR_2}\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - q\right)\right] + \delta N_{12}\left[\frac{2(1 + \nu)N_{12}}{C^*} - \frac{1}{\alpha_2}\frac{\partial u_1}{\partial \xi_2}\right.$$

$$\left. + \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} - \frac{1}{\alpha_1}\frac{\partial u_2}{\partial \xi_1} + \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\right] + \delta M_{11}\left[\frac{M_{11} - \nu M_{22}}{D^*}\right.$$

$$\left. - \frac{1}{\alpha_1}\frac{\partial \beta_1}{\partial \xi_1} - \frac{\beta_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{1}{(h + t)R_1}\frac{k}{\alpha_1\alpha_2}\right]$$

$$+ \delta M_{22}\left[\frac{M_{22} - \nu M_{11}}{D^*} - \frac{1}{\alpha_2}\frac{\partial \beta_2}{\partial \xi_2} - \frac{\beta_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\right.$$

$$\left. + \frac{1}{(h + t)R_2}\frac{k}{\alpha_1\alpha_2}\right] + \delta M_{12}\left[\frac{2(1 + \nu)M_{12}}{D^*} - \frac{1}{\alpha_2}\frac{\partial \beta_1}{\partial \xi_2}\right.$$

$$\left. + \frac{\beta_1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} - \frac{1}{\alpha_1}\frac{\partial \beta_2}{\partial \xi_1} + \frac{\beta_2}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\right] + \delta Q_1\left[\frac{Q_1}{(h + t)G_c} + \frac{u_1}{R_1}\right.$$

$$\left. - \beta_1 - \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1}\right] + \delta Q_2\left[\frac{Q_2}{(h + t)G_c} + \frac{u_2}{R_2} - \beta_2 - \frac{1}{\alpha_2}\frac{\partial w}{\partial \xi_2}\right]$$

$$+ \delta\sigma_{\zeta m}\left[\frac{h + t}{E_c}\sigma_{\zeta m} + \frac{k}{\alpha_1\alpha_2}\right]\Bigg\}\alpha_1\alpha_2\, d\xi_1\, d\xi_2 = 0 \tag{53}$$

As all nine variations in equation (53) are independent of each other, it follows that the contents of all nine brackets in equation (53) must vanish separately. Thus the following nine stress-strain relations are obtained for the sandwich shell, indicating with an asterisk those which appear in final form,

$$\frac{N_{11}}{C^*}\left(1 + \frac{(h + t)C^*}{12E_cR_1^2}\right) - \frac{N_{22}}{C^*}\left(\nu - \frac{(h + t)C^*}{12E_cR_1R_2}\right)$$

$$= \frac{1}{\alpha_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w}{R_1} + \frac{(h + t)q}{12E_cR_1} \tag{54}$$

$$\frac{N_{22}}{C^*}\left(1 + \frac{(h + t)C^*}{12E_cR_2^2}\right) - \frac{N_{11}}{C^*}\left(\nu - \frac{(h + t)C^*}{12E_cR_1R_2}\right)$$

$$= \frac{1}{\alpha_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} + \frac{w}{R_2} + \frac{(h + t)q}{12E_cR_2} \tag{55}$$

$$\frac{2(1 + \nu)}{C^*}N_{12} = \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha_1}\right) + \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2}{\alpha_2}\right) \tag{56*}$$

$$\frac{M_{11} - \nu M_{22}}{D^*} = \frac{1}{\alpha_1}\frac{\partial \beta_1}{\partial \xi_1} + \frac{\beta_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} - \frac{k}{\alpha_1\alpha_2}\frac{1}{(h + t)R_1} \tag{57}$$

$$\frac{M_{22} - \nu M_{11}}{D^*} = \frac{1}{\alpha_2}\frac{\partial \beta_2}{\partial \xi_2} + \frac{\beta_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} - \frac{k}{\alpha_1\alpha_2}\frac{1}{(h + t)R_2} \tag{58}$$

$$\frac{2(1 + \nu)}{D^*}M_{12} = \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(\frac{\beta_1}{\alpha_1}\right) + \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{\beta_2}{\alpha_2}\right) \tag{59*}$$

$$\frac{Q_1}{(h + t)G_c} = \beta_1 + \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} - \frac{u_1}{R_1} \tag{60*}$$

$$\frac{Q_2}{(h + t)G_c} = \beta_2 + \frac{1}{\alpha_2}\frac{\partial w}{\partial \xi_2} - \frac{u_2}{R_2} \tag{61*}$$

$$\frac{\sigma_{\zeta_m}}{E_c} = -\frac{k}{\alpha_1\alpha_2}\frac{1}{(h + t)} \tag{62}$$

It may be verified that the meaning of the quantities u_1, u_2, w, β_1, and β_2 is as has been indicated by comparing equations (54) to (61) with the corresponding equations of reference 2 for the homogeneous shell with $E_c = G_c = \infty$.

The system of equations (54) to (62) may be brought into a slightly more concise form as follows: Define the quantities λ_1, λ_2, and λ_{12} by

$$\left.\begin{aligned}
\lambda_1 &= \frac{1}{2}\frac{(h + t)t}{R_1{}^2}\frac{E_f}{E_c} \\
\lambda_2 &= \frac{1}{2}\frac{(h + t)t}{R_2{}^2}\frac{E_f}{E_c} \\
\lambda_{12} &= \frac{1}{2}\frac{(h + t)t}{R_1R_2}\frac{E_f}{E_c}
\end{aligned}\right\} \tag{63}$$

and eliminate k from equations (57) and (58) by means of equation (62) and the equilibrium equation (34). Retain equations (56) and (59) to (61) in the foregoing form and write for equations (54) and (55)

$$\left(1 + \frac{1}{3}\lambda_1\right)N_{11} - \left(\nu - \frac{1}{3}\lambda_{12}\right)N_{22} = C^*\left(\frac{1}{\alpha_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w}{R_1}\right) + \frac{(h + t)C^*}{12E_cR_1}q \tag{64*}$$

$$\left(1 + \frac{1}{3}\lambda_2\right)N_{22} - \left(\nu - \frac{1}{3}\lambda_{12}\right)N_{11} = C^*\left(\frac{1}{\alpha_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} + \frac{w}{R_2}\right) + \frac{(h + t)C^*}{12E_cR_2} q \tag{65}^*$$

Equations (57) and (58) become, if $D^*/(h + t)E_c = \frac{1}{2} t(h + t)E_f/E_c$, according to equation (47),

$$(1 + \lambda_1)M_{11} - (\nu - \lambda_{12})M_{22} = D^*\left(\frac{1}{\alpha_1}\frac{\partial \beta_1}{\partial \xi_1} + \frac{\beta_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2}\right) + \frac{D^*}{E_cR_1} s \tag{66}^*$$

$$(1 + \lambda_2)M_{22} - (\nu - \lambda_{12})M_{11} = D^*\left(\frac{1}{\alpha_2}\frac{\partial \beta_2}{\partial \xi_2} + \frac{\beta_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\right) + \frac{D^*}{E_cR_2} s \tag{67}^*$$

With these last transformations there is obtained a system of equations which is formally equivalent to the corresponding system of equations for the homogeneous shell. The 5 equilibrium equations (25) to (28) and (31) and the 8 stress-strain relations (56), (59), (60), (61), and (64) to (67) are used for the determination of 13 quantities: Five stress resultants N_{11}, N_{22}, N_{12}, Q_1, and Q_2; three stress couples M_{11}, M_{22}, and M_{12}; and five displacements and changes of slope u_1, u_2, w, β_1, and β_2. The quantity $\sigma\zeta_m$ which occurs in the sixth equilibrium equation (equation (34)) may be determined directly, once the shell bending and stretching problem has been solved.

It is seen that the effect of transverse shear deformation enters equations (60) and (61) only and that, when $G_c = \infty$, these equations give the values of the known theory of homogeneous shells without transverse shear deformation (references 2, 3, and 4).

The effect of transverse normal stress deformation enters equations (64) to (67) only. It is seen that it is, in part, responsible for the occurrence of apparent stiffness factors $C^*/(1 + \lambda)$ and $D^*/(1 + \lambda)$. Thus, according to equation (63), the effect of finite E_c is to make the shell more flexible in bending and stretching than it would be with $E_c = \infty$. This effect, however, is present only in curved structures and not in plates and straight beams, as the quantities λ have one or both of the radii of curvature in the denominator. A further effect of finite E_c is occurrence of the

external load terms q and s in the stress-strain relations. Both these effects represent, roughly speaking, what happens to the shape of an element of the composite shell if the length of the core fibers in transverse direction is changed, without any stretching or compressing of the face-membrane elements.

Having derived the general system of equations for the small bending and stretching of sandwich-type shells, it remains to apply these equations to specific problems which may be of interest and to determine the quantitative effect of the terms which are characteristic of the sandwich-type shell. Some of this work is done in part II of the present report, which follows.

It may be stated once more that for these specific applications the five equilibrium equations (25) to (28) and (31) and the eight stress-strain relations (56), (59), (60), (61), and (64) to (67) are used.

II - APPLICATIONS OF GENERAL THEORY

Flat Plates

The problem of the flat plate is considered first in order to show that the results of reference 1 are contained in the present results and in order to solve some problems in the theory of plates which have not been solved in reference 1.

Rectangular plates.- Using notation which is customary in plate theory there is set

$$\left.\begin{array}{llll}
\xi_1 = x & \xi_2 = y & \alpha_1 = \alpha_2 = 1 & R_1 = R_2 = \infty \\
u_1 = u & u_2 = v & \beta_1 = \beta_x & \beta_2 = \beta_y \\
N_{11} = N_x & N_{12} = N_{xy} & N_{22} = N_y & Q_1 = Q_x \\
Q_2 = Q_y & M_{11} = M_x & M_{12} = M_{xy} & M_{22} = M_y \\
p_1 = p_x & p_2 = p_y & m_1 = m_x & m_2 = m_y
\end{array}\right\} \qquad (68)$$

The equilibrium equations (25) to (28) and (31) become

$$\left.\begin{aligned}\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + p_x &= 0\\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_y &= 0\end{aligned}\right\} \tag{69}$$

$$\left.\begin{aligned}\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q &= 0\\ \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + m_x &= 0\\ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + m_y &= 0\end{aligned}\right\} \tag{70}$$

The stress-strain relations (56), (59), (60), (61), and (64) to (67) become

$$\left.\begin{aligned}N_x - \nu N_y &= C^* \frac{\partial u}{\partial x}\\ N_y - \nu N_x &= C^* \frac{\partial v}{\partial y}\\ 2(1 + \nu)N_{xy} &= C^*\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\end{aligned}\right\} \tag{71}$$

$$\left.\begin{aligned}Q_x &= (h + t)G_c\left(\beta_x + \frac{\partial w}{\partial x}\right)\\ Q_y &= (h + t)G_c\left(\beta_y + \frac{\partial w}{\partial y}\right)\end{aligned}\right\} \tag{72}$$

$$\left.\begin{aligned} M_x - \nu M_y &= D^* \frac{\partial \beta_x}{\partial x} \\ M_y - \nu M_x &= D^* \frac{\partial \beta_y}{\partial y} \\ 2(1 + \nu)M_{xy} &= D^*\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) \end{aligned}\right\} \tag{73}$$

As in the small-deflection theory of homogeneous plates, the equations for stretching (equations (69) and (71)) are independent of the remaining equations for transverse bending. Equations (69) and (71) for the stretching are not affected by the elastic properties of the core layers.

Equations (70), (72), and (73) have been treated in reference 1 by means of a stress function ψ, which, together with the deflection w, was taken as one of two basic variables. In what follows an alternate treatment is given, in which the problem is reduced to three simultaneous equations for the quantities β_x, β_y, and w. On the basis of these three simultaneous equations a problem not considered in reference 1 is treated, namely, the bending of a rectangular plate which is simply supported on all four edges.[7]

To reduce equations (70), (72), and (73) to three simultaneous equations for β_x, β_y, and w, first a quantity ω is defined by

$$\omega = \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} \tag{74}$$

Introducing equation (72) into the first of equations (70), in view of equation (74), there is obtained

$$\omega + \nabla^2 w = - q/(h + t)G_c \tag{75}$$

Next, Q_x, M_x, and M_{xy} are taken from equations (72) and (73) and the result is substituted in the second of equations (70). This gives, after slight transformations,

[7]This same problem has also been solved by L. H. Donnell by a method which differs from the one employed here. (See reference 5 where the case of the homogeneous plate is considered.)

$$\frac{D^*}{1 + \nu} \nabla^2\beta_x - 2(h + t)G_c\beta_x + \frac{\partial}{\partial x}\left[\frac{D^*\omega}{1 - \nu} - 2(h + t)G_c w\right] + m_x = 0 \qquad (76)$$

In an analogous manner the following further equation is obtained:

$$\frac{D^*}{1 + \nu} \nabla^2\beta_y - 2(h + t)G_c\beta_y + \frac{\partial}{\partial y}\left[\frac{D^*\omega}{1 - \nu} - 2(h + t)G_c w\right] + m_y = 0 \qquad (77)$$

In order to solve equations (75) to (77) two equations are next obtained involving w and ω only. Differentiating equation (76) with respect to x and equation (77) with respect to y and adding the two resultant equations, in view of equation (74), gives

$$\frac{2D^*}{1 - \nu^2} \nabla^2\omega - 2(h + t)G_c\left(\omega + \nabla^2 w\right) + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} = 0$$

and, making use of equation (75),

$$\nabla^2\omega = -\frac{1}{D}\left[q + \frac{1}{2}\left(\frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}\right)\right] \qquad (78)$$

The following procedure may now be carried out: (a) Solve equation (78) for ω, (b) with this value of ω solve equation (75) for w, (c) substitute ω and w in equations (76) and (77) and solve for β_x and β_y, and (d) eliminate extraneous terms in β_x and β_y by considering equation (74).

Before deriving the solution of a problem along these lines, the explicit differential equation for w which follows by combining equations (75) and (78) may be given

$$\nabla^2\nabla^2 w = \frac{1}{D}\left[q + \frac{1}{2}\left(\frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}\right)\right] - \frac{\nabla^2 q}{(h + t)G_c} \qquad (79)$$

Note that the effect of transverse shear occurs on the right side of the equation only. In order to compare the magnitude of the q terms on the right of equation (79), assume that relevant changes of q occur over distances of order l (where l may or may not be a representative diameter of the plate). Then, as order-of-magnitude relations, there results

$$\left.\begin{aligned} \frac{q}{D^*} &= 0\left(\frac{q}{E_f t h^2}\right) \\ \frac{\nabla^2 q}{(h + t)G_c} &= 0\left(\frac{q}{l^2 h G_c}\right) \end{aligned}\right\} \tag{80}$$

From equation (80), it follows that transverse shear ceases to be a secondary effect as soon as l is of order $\sqrt{ht}\ \sqrt{E_f/G_c}$ or of smaller order.

Bending of rectangular plate with simply supported edges.- The edges of the plate are assumed to be at $x = 0,a$ and $y = 0,b$ and along these edges moments and deflections are assumed to vanish. Further,

$$\left.\begin{aligned} m_x &= m_y = 0 \\ q &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \lambda_m x \sin \mu_n y \end{aligned}\right\} \tag{81}$$

where

$$\left.\begin{aligned} \lambda_m &= m\pi/a \\ \mu_n &= n\pi/b \end{aligned}\right\} \tag{82}$$

From equation (78), it follows that

$$\omega = \frac{1}{D} \sum \sum \frac{q_{mn}}{\lambda_m^2 + \mu_n^2} \sin \lambda_m x \sin \mu_n y + \omega_h \tag{83}$$

where ω_h is a harmonic function. Putting equation (83) into equation (75),

$$\nabla^2 w = -\sum\sum q_{mn}\left[\frac{1/D}{\lambda_m^2 + \mu_n^2} + \frac{1}{(h + t)G_c}\right] \sin \lambda_m x \sin \mu_n y - \omega_h$$

which is integrated to

$$w = \frac{1}{D}\sum\sum \frac{q_{mn}}{\left(\lambda_m^2 + \mu_n^2\right)^2}\left[1 + \frac{D\left(\lambda_m^2 + \mu_n^2\right)}{(h + t)G_c}\right] \sin \lambda_m x \sin \mu_n y + w_h \quad (84)$$

where w_h is the general solution of $\nabla^2 w_h = -\omega_h$. It is to be expected and may be shown explicitly that for the plate which is simply supported all around $w_h = \omega_h = 0$ and, as in the Navier solution for the plate without transverse shear deformation, the particular integral is the complete solution of the problem.

Equation (84) may be rewritten in the more explicit form

$$w = \frac{a^4}{D\pi^4}\sum\sum \frac{q_{mn}\left\{1 + \frac{\pi^2}{2}\frac{E_f}{(1 - \nu^2)G_c}\frac{(h + t)t}{a^2}\left[m^2 + n^2\left(a^2/b^2\right)\right]\right\}}{\left[m^2 + n^2\left(a^2/b^2\right)\right]^2}$$

$$\times \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \quad (85)$$

When $G_c = \infty$, equation (85) reduces to Navier's solution w_N. Equation (85) is more readily interpreted by means of the ratio w/w_N of deflection with and without transverse shear deformation. On the basis of equation (85), there may be obtained the following equation (86),[8]

[8]Setting $\pi^2/2(1 - \nu^2) = 5.4$ and $(E_f/G_c)t(h + t)/a^2 = \beta$, equation (86) takes on a form which contains as a special case the result of equation (18) of reference 5.

$$\frac{w}{w_N} = \frac{w_N - \dfrac{D\nabla^2 w_N}{(h + t)G_c}}{w_N} = 1 - \frac{(h + t)tE_f}{2(1 - \nu^2)G_c} \frac{\nabla^2 w_N}{w_N}$$

$$= 1 + \frac{\pi^2}{2} \frac{E_f}{(1 - \nu^2)G_c} \frac{(h + t)t}{a^2} \frac{\sum\sum \dfrac{q_{mn} \sin m\pi x/a \sin n\pi y/b}{m^2 + n^2(a^2/b^2)}}{\sum\sum \dfrac{q_{mn} \sin m\pi x/a \sin n\pi y/b}{\left[m^2 + n^2(a^2/b^2)\right]^2}} \tag{86}$$

For the case of a uniform load intensity q = Constant and for the center of the plate ($x = a/2$, $y = b/2$) equation (86) becomes

$$\frac{w}{w_N} = 1 + \frac{\pi^2}{2} \frac{E_f}{(1 - \nu^2)G_c} \frac{(h + t)t}{a^2} \frac{\sum\sum \dfrac{\sin m\pi/2 \sin n\pi/2}{mn\left[m^2 + n^2(a^2/b^2)\right]}}{\sum\sum \dfrac{\sin m\pi/2 \sin n\pi/2}{mn\left[m^2 + n^2(a^2/b^2)\right]^2}} \tag{87}$$

The ratio of the series is 1.98, when $a/b = 1$, and the ratio of the series is 1.11 when $a/b = 1/2$.

For the case of a concentrated load at the center of the plate the deflection ratio at the point of load application assumes the form

$$\frac{w}{w_N} = 1 + \frac{\pi^2}{2} \frac{E_f}{(1 - \nu^2)G_c} \frac{(h + t)t}{a^2} \frac{\sum\sum \dfrac{(\sin m\pi/2 \sin n\pi/2)^2}{m^2 + n^2(a^2/b^2)}}{\sum\sum \dfrac{(\sin m\pi/2 \sin n\pi/2)^2}{\left[m^2 + n^2(a^2/b^2)\right]^2}} \tag{88}$$

Now it is easily shown that the numerator series in equation (88) does not converge and consequently $w/w_N = \infty$ in this case. A more detailed consideration shows that in any plate theory which takes transverse shear deformation into account the deflection under the point of application of a concentrated load must become infinite in contrast to what happens when transverse shear deformation is not taken into

account. This difference, of course, vanishes as soon as the load intensity becomes finite, and then the theory with transverse shear deformation taken into account is more accurate than the theory which does not take into account this effect.

For the sake of numerical illustration take again the square plate $(a/b = 1)$ with uniform load distribution. According to equation (87), the deflection at the center is increased because of transverse shear by the factor

$$\frac{w}{w_N} = 1 + 9.7 \frac{E_f}{G_c} \frac{(h + t)t}{a^2} \tag{87a}$$

Take $h = 1.0$ inch, $t = 0.1$ inch, $a = 10$ inches, $E_f/G_c = 200$, and $\nu = 1/3$. Then, according to equation (87a), $w/w_N = 1 + 2.3$, so that in this case the deflection with transverse shear is more than three times the deflection when shear deformation in the core is neglected.

Returning now to equation (84) for w and equation (83) for ω and substituting these two equations in equations (76) and (77) in order to determine the changes of slope β_x and β_y, after slight transformations there results

$$\left.\begin{aligned} \beta_x &= -\frac{1}{D} \sum \sum \frac{q_{mn}\lambda_m}{\left(\lambda_m^2 + \mu_n^2\right)^2} \cos \lambda_m x \sin \mu_n y \\ \beta_y &= -\frac{1}{D} \sum \sum \frac{q_{mn}\mu_n}{\left(\lambda_m^2 + \mu_n^2\right)^2} \sin \lambda_m x \cos \mu_n y \end{aligned}\right\} \tag{89}$$

Equations (89) are remarkable for the reason that they are not affected by transverse shear deformability. According to equations (73), the same is then true of the bending and twisting couples M_x, M_y, and M_{xy}. It is not easy to see why, in this statically indeterminate problem, the magnitude of the internal forces, as well as that of the deflections, does not depend on the elastic properties of the core. The analysis, however, shows that the distributions of M_x, M_y, and M_{xy}, and therewith of Q_x and Q_y, remain the same as those obtained under the assumption that $G_c = \infty$. In this connection the following remark may be made.

Evidently the following three boundary conditions, $w = M_x = \beta_y = 0$ along the edges $x = 0,a$, have been satisfied. In order that the last of these three conditions be satisfied there are necessarily nonvanishing edge values of the twisting couples M_{xy}. The same is true in the theory without transverse shear deformation, where, however, no alternative possibility exists, as in that theory only the boundary conditions $w = M_x = 0$ are relevant. For the present system of equations three boundary conditions must be formulated for every plate edge. Thus, it is possible although mathematically complicated to solve the problem of the rectangular simply supported plate with the edge condition $\beta_y = 0$ replaced by the condition $M_{xy} = 0$. In that case, which will not be pursued here, there evidently will be a distribution of internal stresses which is modified by the effect of transverse shear deformation.

Cylindrical bending of plates.– As a further relatively simple example of application of equations (70), (72), and (73) problems are considered for which

$$\left.\begin{aligned} &\partial(\ )/\partial y = 0 \\ &\partial(\ )/\partial x = d(\ )/dx = (\ )' \\ &M_{xy} = Q_y = m_y = \beta_y = 0 \\ &M_y = \nu M_x \end{aligned}\right\} \qquad (90)$$

and where consequently the problem reduces to the following system of equations:[9]

$$\left.\begin{aligned} &Q_x' + q = 0 \\ &M_x' - Q_x + m_x = 0 \\ &Q_x = (h + t)G_c(\beta_x + w') \\ &(1 - \nu^2)M_x = D^*\beta_x' \end{aligned}\right\} \qquad (91)$$

[9]Note that in order to obtain the problem of the sandwich beam from equations (90) and (91) the only changes which are necessary amount to setting $\nu = 0$ in equations (91).

To set into evidence the effect of finite values of G_c in equation (91), the following system of equations is deduced from equation (91):

$$Dw^{(IV)} = q + m_x' - \frac{Dq''}{(h + t)G_c} \tag{92}$$

$$M_x = -Dw'' - \frac{Dq}{(h + t)G_c} \tag{93}$$

$$Q_x = -Dw''' - \frac{Dq'}{(h + t)G_c} + m_x \tag{94}$$

$$\beta_x = -w' + \frac{Q_x}{(h + t)G_c} \tag{95}$$

Solutions to the following problems are listed:

(1) Simply supported plate of span l carrying a load $q = q_0 \cos \pi x/l$. Boundary conditions: $w(\pm l/2) = M_x(\pm l/2) = 0$.

$$w = \frac{q_0}{D}\left[1 + \frac{\pi^2}{2} \frac{E_f}{(1 - \nu^2)G_c} \frac{(h + t)t}{l^2}\right] \frac{\cos \pi x/l}{(\pi/l)^4} \qquad (96)^{10}$$

As the problem is statically determinate as far as moment and force are concerned there is no modification of M_x and Q_x due to the finite value of G_c.

[10]The factor in brackets may again be written in the form $1 + 5.4\beta$, with $\beta = (E_f/G_c)\Big/\left[(h + t)t/l^2\right]$, using the notation suggested in reference 5.

(2) Simply supported plate of span l carrying a uniform load $q = q_0$.

$$w = \frac{q_0 l^4}{D16}\left\{\frac{1}{24}\left[\left(\frac{x}{l/2}\right)^4 - 1\right] - \frac{1}{4}\left[1 + \frac{8D}{(h + t)G_c l^2}\right]\left[\left(\frac{x}{l/2}\right)^2 - 1\right]\right\} \tag{97}$$

From this for the center deflection,

$$w(0) = \frac{5}{384}\frac{q_0 l^4}{D}\left[1 + \frac{24}{5}\frac{E_f}{(1 - \nu^2)G_c}\frac{(h + t)t}{l^2}\right] \tag{98}$$

It is seen that the correction factor for the center deflection is almost the same as that for the cosine load curve (equation (96)), the only difference being a change of the factor $\pi^2/2 = 4.93$ into $24/5 = 4.80$, that is, a reduction of the shear correction factor by at most 3 percent is present.[11]

(3) Built-in plate of span l carrying a uniform load $q = q_0$. The boundary conditions are: $w(\pm l/2) = \beta_x(\pm l/2) = 0$ (and not $w'(\pm l/2) = 0$).

$$w = \frac{q_0 l^4}{384D}\left\{\left[\left(\frac{x}{l/2}\right)^4 - 1\right] - 2\left[1 + \frac{24D}{(h + t)G_c l^2}\right]\left[\left(\frac{x}{l/2}\right)^2 - 1\right]\right\} \tag{99}$$

From this there follows for the center deflection,

$$w(0) = \frac{q_0 l^4}{384D}\left[1 + 24\frac{E_f}{(1 - \nu^2)G_c}\frac{(h + t)t}{l^2}\right] \tag{100}$$

Comparison of equations (100) and (98) shows that for the built-in plate the effect of transverse shear deformation is very much more

[11]Note that according to equation (87a) the shear correction factor for the square plate of width $a = l$ is more than twice as large as the shear correction factor for the plate strip of width l.

pronounced than it is for the simply supported plate, a factor 24/5 in the latter case being replaced by a factor 24 in the former case.[12] As a further result in this problem of the built-in plate, by putting equation (99) into equation (93), it is found that the moment function M_x does not contain any terms depending on the effect of transverse shear deformation. This again is somewhat surprising as in this case it is not possible to determine the moment function by statics alone.[13]

Circular plates; rotational symmetry.- As no examples of solutions of circular sandwich-plate problems have as yet been published and as it is of some interest to determine in which way the shear correction factors change in going from a problem for the plate strip to the corresponding problem for the circular plate, the equations for axisymmetrical transverse bending of circular plates are briefly discussed.

Polar coordinates r,θ are introduced and notation which is customary in plate theory is used. As a consequence of equations (70), (72), and (73), the following system of equations is obtained:

$$\left.\begin{aligned} &\frac{drQ_r}{dr} + rq = 0 \\ &\frac{drM_r}{dr} - M_\theta - rQ_r + rm_r = 0 \end{aligned}\right\} \tag{101}$$

$$Q_r = (h + t)G_c\left(\beta_r + dw/dr\right) \tag{102}$$

[12]A somewhat similar percentage increase must take place in going from equation (86) for the rectangular plate with all four edges simply supported to a formula (which has not yet been derived) for the rectangular plate with all four edges built in.

[13]As a problem where the moment distribution is in fact dependent on the effect of transverse shear there may be mentioned the problem of the cylindrically bent plate with both ends built in, which carries a load $q_1 = q_1x$ instead of the load $q_0 = q_0$. This problem also may be solved by means of equations (92) to (95).

$$\left.\begin{aligned} M_r - \nu M_\theta &= D^* \, d\beta_r/dr \\ M_\theta - \nu M_r &= D^* \beta_r/r \end{aligned}\right\} \tag{103}$$

According to equation (79), the equation for the deflection w will be

$$D\nabla^2\nabla^2 w = q + \frac{1}{2}\frac{1}{r}\frac{drm_r}{dr} - \frac{\nabla^2 q}{(h + t)G_c} \tag{104}$$

where $\nabla^2 = (1/r)d\left[r \, d(\ )/dr\right]/dr$. Having found w by means of equation (104), β_r may be determined from

$$\beta_r = -\frac{dw}{dr} + \frac{Q_r}{(h + t)G_c} = -\frac{dw}{dr} + \frac{1}{r}\int rq \, dr \tag{105}$$

and therewith M_r and M_θ are obtained from equations (103).

In the present problem it seems to be somewhat more convenient to proceed as follows: Combine equations (101) and (103) to obtain as equation for the change of slope β_r,

$$D\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{dr\beta_r}{dr}\right)\right] = q - \frac{1}{r}\frac{drm_r}{dr} \tag{106}$$

Having β_r, M_r and M_θ are found from equation (103) and Q_r, from the second of equations (101),

$$\left.\begin{aligned} M_r &= D\left(\frac{d\beta_r}{dr} + \frac{\nu}{r}\beta_r\right) \\ M_\theta &= D\left(\frac{1}{r}\beta_r + \nu\frac{d\beta_r}{dr}\right) \end{aligned}\right\} \tag{107}$$

$$Q_r = D \frac{d}{dr}\left(\frac{1}{r} \frac{dr\beta_r}{dr}\right) + m_r \tag{108}$$

Finally, with this value of Q_r, w is found by integrating equation (102),

$$w = -\int \beta_r \, dr + \frac{D}{(h + t)G_c}\left(\frac{1}{r} \frac{dr\beta_r}{dr}\right) + \frac{\int m_r \, dr}{(h + t)G_c} \tag{109}$$

<u>Deflection of circular plate with built-in edge</u>.— The bending is now considered of a plate with transverse load $q = q_n(r/a)^n$ and with $m_r = 0$. First, from equation (106),

$$D\beta_r = c_1 \frac{r}{a} + c_2\left(\frac{r}{a}\right)^{-1} + c_3 \frac{r}{a} \log_e \frac{r}{a} + \frac{q_n a^3}{(n + 4)(n + 2)^2}\left(\frac{r}{a}\right)^{n+3} \tag{110}$$

Attention is restricted to complete plates with no concentrated load at the center, and consequently it is necessary to set $c_2 = c_3 = 0$ in equation (110). This gives

$$D\beta_r = c_1 \frac{r}{a} + \frac{q_n a^3}{(n + 4)(n + 2)^2}\left(\frac{r}{a}\right)^{n+3} \tag{110a}$$

Putting equation (110a) into equation (109), there results for the transverse deflection w

$$\begin{aligned} D \frac{w}{a} = &-\left[\frac{c_1}{2}\left(\frac{r}{a}\right)^2 + \frac{q_n a^3}{(n + 4)(n + 2)^2}\left(\frac{r}{a}\right)^{n+4} + c_4\right] \\ &+ \frac{D}{a^2(h + t)G_c}\left[2c_1 + \frac{q_n a^3}{(n + 2)^2}\left(\frac{r}{a}\right)^{n+2}\right] \end{aligned} \tag{111}$$

Taking the case of a plate of radius a with built-in edge, that is, with the boundary conditions

$$\beta_r(a) = w(a) = 0 \tag{112}$$

there results

$$D\beta_r = \frac{q_n a^3}{(n + 4)(n + 2)^2}\left[\left(\frac{r}{a}\right)^{n+3} - \frac{r}{a}\right] \tag{113}$$

and

$$Dw = -\frac{q_n a^4}{(n + 2)^2}\left\{\frac{(r/a)^{n+4} - 1}{(n + 4)^2} - \frac{1}{2}\,\frac{(r/a)^2 - 1}{n + 4} - \frac{D}{a^2(h + t)G_c}\left[\left(\frac{r}{a}\right)^{n+2} - 1\right]\right\} \tag{114}$$

From equation (114) there follows for the deflection at the center of the plate

$$Dw(0) = \frac{q_n a^4}{2(n + 2)(n + 4)^2}\left[1 + \frac{(n + 4)^2}{n + 2}\,\frac{E_f}{(1 - \nu^2)G_c}\,\frac{(h + t)t}{a^2}\right] \tag{115}$$

Consider the following special cases:

(1) Uniform load distribution $q_n = q_0$. From equation (115), it follows for the ratio of deflection with and without transverse shear deformation that

$$\frac{w(0)}{\left[w(0)\right]_{G_c=\infty}} = 1 + 8\,\frac{E_f}{(1 - \nu^2)G_c}\,\frac{(h + t)t}{a^2} \tag{116}$$

Equation (116) may be compared with equation (100) for the deflection of the infinite plate strip of width l with built-in edges. Setting $l = 2a$, it is seen that, while the transverse shear correction factor for the strip has a value 6, the corresponding factor for the circular plate is 8. This is consistent with the earlier comparison between the simply supported strip and the simply supported square plate, except that there the change is from 4.8 to 9.7.

(2) Linearly increasing load distribution $q = q_1 r/a$. From equation (115), it follows that

$$\frac{w(0)}{\left[w(0)\right]_{G_c=\infty}} = 1 + \frac{25}{3} \frac{E_f}{(1 - \nu^2)G_c} \frac{(h + t)t}{a^2} \tag{117}$$

showing that the correction effect is only slightly greater than in the case of the uniform load distribution.

(3) Load increasing linearly from edge to center, $q = q_0 + q_1(r/a)$. $(q_1 = -q_0)$. From equation (115), it follows by superposition that

$$\begin{aligned} Dw(0) &= \frac{q_0 a^4}{2 \times 2 \times 16}\left[1 + 8 \frac{E_f}{(1 - \nu^2)G_c} \frac{(h + t)t}{a^2}\right] \\ &\quad - \frac{q_0 a^4}{2 \times 3 \times 25}\left[1 + \frac{25}{3} \frac{E_f}{(1 - \nu^2)G_c} \frac{(h + t)t}{a^2}\right] \end{aligned}$$

$$Dw(0) = \frac{43 q_0 a^4}{32 \times 150}\left[1 + \frac{3000}{387} \frac{E_f}{(1 - \nu^2)G_c} \frac{(h + t)t}{a^2}\right] \tag{118}$$

Comparing the factor $3000/387 = 7.76$ which occurs in equation (118) with the corresponding factors 8 and 8.33 in equations (116) and (117) it is seen that, in the foregoing three problems at least, there is little difference between the transverse shear stress correction factors in the case of three different loading conditions for the circular, clamped-edge plate. The fact that this agreement should not be expected to hold generally follows again by considering the case of a point load at the center of the plate, for which the shear correction factor would again be infinite.

The examples of this section should be augmented by the solution for the circular plate of radius a, which carries a load distributed uniformly over a smaller circle which is concentric with the boundary of the plate.

Circular Rings

As the simplest example of a curved sandwich structure there are considered in this section stresses and deformations of circular rings in their own plane. As was found in the general developments of part I of this report, in a curved sandwich structure there will be the effect of both transverse shear and normal stress deformation.[14]

There are set for the relevant coordinates and variables

$$\left.\begin{array}{ll}
\xi_1 = a\theta & \alpha_1 = 1 \\
R_1 = a & u_1 = v \\
\beta_1 = \beta & N_{11} = N \\
Q_1 = Q & M_{11} = M \\
p_1 = p & m_1 = m \\
\multicolumn{2}{c}{\partial(\ )/\partial\xi = d(\ )/a\ d\theta = (\ )'/a} \\
\multicolumn{2}{c}{\lambda_1 = \lambda = \frac{1}{2}\left[(h + t)t/a^2\right]\left(E_f/E_c\right)}
\end{array}\right\} \qquad (119)$$

The equilibrium equations (25) to (28), (31), and (34) reduce to the following equations:

$$\left.\begin{array}{l}
N' + Q + ap = 0 \\
Q' - N + aq = 0
\end{array}\right\} \qquad (120)$$

[14]The effect of transverse shear stress deformation on homogeneous circular rings has been considered by L. Beskin in reference 6.

$$\left.\begin{aligned} M' - aQ + am &= 0 \\ \sigma_{\zeta m} &= s - [M/(h + t)a] \end{aligned}\right\} \tag{121}$$

The stress-strain relations (56), (59), (60), (61), and (64) to (67) reduce to the following equations:

$$\left(1 + \frac{1}{3}\lambda\right)N = \frac{1}{a} C^* \left[v' + w + \frac{(h + t)q}{12E_c}\right] \tag{122}$$

$$Q = (h + t)G_c\left[\beta + \frac{1}{a}(w' - v)\right] \tag{123}$$

$$(1 + \lambda)M = \frac{1}{a} D^*(\beta' + s/E_c) \tag{124}$$

The load terms p, q, m, and s are given, according to equations (19) to (22), by

$$\left.\begin{aligned} p &= \left(1 + \frac{h + t}{2a}\right)p_u + \left(1 - \frac{h + t}{2a}\right)p_l \\ q &= \left(1 + \frac{h + t}{2a}\right)q_u + \left(1 - \frac{h + t}{2a}\right)q_l \\ m &= \frac{h + t}{2}\left[\left(1 + \frac{h + t}{2a}\right)p_u - \left(1 - \frac{h + t}{2a}\right)p_l\right] \\ s &= \frac{1}{2}\left[\left(1 + \frac{h + t}{2a}\right)q_u - \left(1 - \frac{h + t}{2a}\right)q_l\right] \end{aligned}\right\} \tag{125}$$

Ring sector acted upon by end bending moments.— As a first problem on circular rings, which illustrates the effect of transverse normal stress deformation, there is taken this basic case for which, as is known, there must be the same stress distribution at all sections θ = Constant of the ring.

According to equations (120) and (121),

$$\left.\begin{aligned} N &= Q = 0 \\ M &= M_0 \\ \sigma_{\zeta_m} &= -M_0/(h + t)a \end{aligned}\right\} \tag{126}$$

Equations (122) to (124) become

$$\left.\begin{aligned} v' + w &= 0 \\ \beta + (w' - v)/a &= 0 \end{aligned}\right\} \tag{127}$$

$$(1 + \lambda)M_0 = D^*\beta'/a \tag{128}$$

The significant result of this consideration is contained in equation (128), which may be written in the alternate form

$$M = M_0 = \frac{D^*}{a}\frac{\beta'}{1 + \lambda} = \frac{D^*}{1 + \frac{1}{2}\frac{(h + t)t}{a^2}\frac{E_f}{E_c}}\frac{w'' + w}{-a^2} \tag{129}$$

Thus, in this case of pure bending the transverse flexibility of the core is responsible for a reduction of the bending stiffness factor $D^* = \frac{1}{2}t(h + t)^2E_f$ which is obtained exactly when $E_c = 0$ and practically when E_c is of the same order of magnitude as E_f. Equation (129) shows that the reduction of D^* is significant whenever E_c is so small that the ratio E_c/E_f is of the same order of magnitude as the ratio $(h + t)t/a^2$.

As a numerical example take the following values: $h = 0.9$ inch, $t = 0.05$ inch, $a = 20$ inches, and $E_f/E_c = 1000$, for which

$$\frac{1}{2}\frac{(h + t)t}{a^2}\frac{E_f}{E_c} = \frac{1}{2} \times \frac{0.95 \times 0.05}{400} \times 1000 = 0.0595$$

indicating a reduction in bending stiffness of about 6 percent. Changing a from 20 inches to 10 inches changes the effect from 6 percent to 24 percent. Changing E_f/E_c from 1000 to 2000 increases the effect from 6 percent to 12 percent. Altogether it may be said that this effect is of noticeable magnitude for some geometrically reasonable structures when the modulus ratio E_f/E_c is of the order 1000 or more. Assuming aluminum face layers with $E_f = 10^7$ psi, this means that $E_c \approx 10^4$ psi, which is well within the range of some present-day core-layer materials.

Comparing equation (129) with the earlier formulas for the effect of transverse shear stress deformation, for instance with equation (116) in which a represents the plate radius and observing that $G_c \approx \frac{1}{2} E_c$, it is seen that the correction terms are of the same form, the difference being an appreciably larger numerical factor in the expression representing the shear effect.

Closed circular ring acted upon by uniform radial load.- Having rotational symmetry, $d/d\theta = 0$ and $v = \beta = 0$. Also set $p = m = 0$. The remaining equations permit the determination of the stresses in the face and core layers in a way which depends on the extent to which the load is applied to the outer (upper) and inner (lower) face membranes. Equation (12) becomes

$$N = aq \tag{130}$$

From equations (121), it follows that

$$\sigma_{\zeta_m} = s - \left[M/(h + t)a\right] \tag{131}$$

The stress-strain relations (122) to (124) give

$$\frac{w}{a} = \frac{1}{C^*}\left(1 + \frac{\lambda}{3}\right)aq - \frac{(h + t)q}{12E_c a} = \frac{aq}{C^*}\left[1 + \frac{\lambda}{3} - \frac{(h + t)C^*}{12E_c a^2}\right] = \frac{aq}{C^*} \tag{132}$$

and

$$M = \frac{D^*}{1 + \lambda}\,\frac{s}{aE_c} \tag{133}$$

A closed circular ring subjected to a uniform radial load distribution q is stressed not only by a uniform axial force $N = aq$, as would be expected, but in addition is stressed by a uniform bending moment M, the magnitude of which is given by equation (133). The explanation of this result is that for a ring with relatively soft core the circumferential stress distribution depends on the extent to which the external radial load is applied to the inner and outer forces, respectively. Roughly speaking, for a sufficiently flexible core layer the load q_u goes predominantly into the outer face layer, while the load q_l goes predominantly into the inner face layer.

According to equations (46), in the present case for the stresses in the two face layers,

$$\left.\begin{aligned} N_u &= \frac{1}{2}N + \frac{1}{h + t}M \\ N_l &= \frac{1}{2}N - \frac{1}{h + t}M \end{aligned}\right\} \tag{134}$$

According to equations (130) and (133) and in view of the definitions of D^* and λ, this may be written

$$\left.\begin{aligned} N_u &= \frac{a}{2}\left(q + \frac{2\lambda s}{1 + \lambda}\right) \\ N_l &= \frac{a}{2}\left(q - \frac{2\lambda s}{1 + \lambda}\right) \end{aligned}\right\} \tag{135}$$

Combining next equations (131) and (133), for the transverse normal stress in the core layer, the following expression is obtained:

$$\sigma_{\zeta_m} = \frac{s}{1 + \lambda} \tag{136}$$

For a specific example assume that the radial load is applied entirely to the inner face of the ring so that $q_u = 0$ and, according to equation (125),

$$\left.\begin{aligned} q &= \left(1 - \frac{h + t}{2a}\right)q_l \\ s &= -\frac{1}{2}\left(1 - \frac{h + t}{2a}\right)q_l \end{aligned}\right\} \tag{137}$$

With q and s given by equations (137), equations (135) and (136) become

$$\left.\begin{aligned} N_u &= \frac{1}{2}\left(1 - \frac{h + t}{2a}\right)\frac{aq_l}{1 + \lambda} \\ N_l &= \frac{1}{2}\left(1 - \frac{h + t}{2a}\right)\frac{(1 + 2\lambda)aq_l}{1 + \lambda} \end{aligned}\right\} \tag{138}$$

$$\sigma_{\zeta_m} = -\frac{1}{2}\left(1 - \frac{h + t}{2a}\right)\frac{q_l}{1 + \lambda} \tag{139}$$

It is seen that the flexibility of the core layer increases the circumferential stress in the loaded face layer in the ratio $(1 + 2\lambda)/(1 + \lambda)$ and decreases it in the unloaded face layer in the ratio $1/(1 + \lambda)$, where λ is defined by equation (119), compared with the equal values of these stresses when $E_c = \infty$.

Considering once more the numerical data under the section entitled "Ring sector acted upon by end bending moments," it is found, for instance, that the stress in the inner face layer may be about 6 or 12,

or 24 percent higher than the corresponding stress calculated without taking into account the transverse flexibility of the core layer.

Ring sector acted upon by radial loads q_u and q_l, uniform in circumferential direction and with vanishing resultant q.— Again it is assumed that $d(\ )/d\theta = 0$, $m = p = 0$ and now in addition that $q = 0$, so that, according to equation (125), the only nonvanishing load term is s. Further, it is assumed that the ends $\theta = \pm\alpha$ of the ring sector are free of stress, that is, $N(\pm\alpha) = Q(\pm\alpha) = M(\pm\alpha) = 0$. The ordinary theory of circular rings would then indicate the absence of deformations in the entire ring. In the present case there is found a type of deformation peculiar to the sandwich ring, which may perhaps be compared to the action of a Bourdon gage.

Solving first equations (120) and (121) and satisfying the end conditions of the ring sector,

$$\left.\begin{aligned} N = Q = M = 0 \\ \sigma_{\zeta_m} = s \end{aligned}\right\} \tag{140}$$

The stress-strain relations (122) to (124) are then

$$\left.\begin{aligned} v' + w = 0 \\ a\beta + w' - v = 0 \\ \beta' = -s/E_c \end{aligned}\right\} \tag{141}$$

Assuming s independent of θ, from equation (141) there is obtained by integration, with constants of integration A_1, A_2, and A_3,

$$\left.\begin{aligned} \beta &= -\frac{s}{E_c}\,\theta + A_1 \\ v &= -a\frac{s}{E_c}\,\theta + A_1 a + A_2 \cos\theta + A_3 \sin\theta \\ w &= a\frac{s}{E_c} + A_2 \sin\theta - A_3 \cos\theta \end{aligned}\right\} \tag{142}$$

As a specific example consider a complete ring, slitted radially at the section $\theta = \pi$, so that $\alpha = \pi$. Prescribe furthermore the symmetry conditions $\beta(0) = v(0) = w(0) = 0$. Under these conditions there is obtained from equation (142)

$$\left.\begin{aligned} E_c\beta &= -s\theta \\ E_c v &= -as(\theta - \sin\theta) \\ E_c w &= as(1 - \cos\theta) \end{aligned}\right\} \tag{143}$$

From equations (143), it follows that the radial slit, which is of zero width before the loads q_u and q_l are applied, opens under the action of the loads to a width given by

$$v(-\pi) - v(\pi) = 2\pi a \frac{s}{E_c} = 2\pi a\left(1 + \frac{h + t}{2a}\right)\frac{q_u}{E_c} \tag{144}$$

For a numerical example take $a = 10$ inches, $h = 1$ inch, $t = 0.05$ inch, $E_c = 10{,}000$ psi, and $q_u = 20$ psi, and obtain

$$v(-\pi) - v(\pi) = 0.132 \text{ inch} \tag{145}$$

The foregoing three examples of ring analysis have been discussed in some detail, because they illustrate relatively simply the effect of transverse normal stress deformation in the theory of curved sandwich structures, without involving at the same time the effect of transverse shear stress deformation.

Bending of semicircular ring by end shear forces.- Now a problem is considered in which both the values of E_c and G_c affect the result of the analysis. In the equilibrium equations (120) and (121) all external load terms are set equal to zero and then, by integration and from the boundary conditions, that is, from

$$\left.\begin{aligned} N\left(\pm\frac{\pi}{2}\right) &= M\left(\pm\frac{\pi}{2}\right) = 0 \\ Q\left(\pm\frac{\pi}{2}\right) &= \pm Q_0 \end{aligned}\right\} \tag{146}$$

the following expressions for N, M, and Q are obtained:

$$\left.\begin{aligned} Q &= Q_0 \sin\theta \\ N &= Q_0 \cos\theta \\ M &= -aQ_0 \cos\theta \end{aligned}\right\} \qquad (147)$$

The stress—strain relations (122) to (124) become

$$\left.\begin{aligned} (1 + \lambda/3)Q_0 \cos\theta &= (C^*/a)(v' + w) \\ Q_0 \sin\theta &= (h + t)G_c\left[\beta + (w' - v)/a\right] \\ -(1 + \lambda)Q_0 a \cos\theta &= (D^*/a)\beta' \end{aligned}\right\} \qquad (148)$$

Integration of the last of equations (148) gives

$$D^*\beta = -a^2(1 + \lambda)Q_0 \sin\theta \qquad (149)$$

where a constant of integration has been eliminated by means of the symmetry condition $\beta(0) = 0$. Substituting equation (149) in the second of equations (148),

$$\begin{aligned} \frac{1}{a}(w' - v) &= Q_0 \sin\theta\left[\frac{1}{(h + t)G_c} + \frac{a^2(1 + \lambda)}{D^*}\right] \\ &= Q_0 \sin\theta\,\frac{a^2}{D^*}\left[1 + \frac{1}{2}\,\frac{(h + t)t}{a^2}\left(\frac{E_f}{E_c} + \frac{E_f}{G_c}\right)\right] \end{aligned} \qquad (150)$$

Simultaneous solution of equation (150) and the first of equations (148) for v and w gives as general expressions for v and w,

$$\left.\begin{aligned} v &= A\theta \cos\theta + A_1 \sin\theta + A_2 \cos\theta \\ w &= A\theta \sin\theta - (A_1 + B)\cos\theta + A_2 \sin\theta \end{aligned}\right\} \qquad (151)$$

where A_1 and A_2 are arbitrary constants of integration and A and B are found to be

$$\left.\begin{aligned} A &= \frac{Q_0 a^3}{2D^*}\left[1 + \frac{1}{2}\frac{(h + t)t}{a^2}\left(\frac{E_f}{E_c} + \frac{E_f}{G_c}\right) + \frac{D^*}{a^2C^*}\left(1 + \frac{\lambda}{3}\right)\right] \\ B &= \frac{Q_0 a^3}{2D^*}\left[1 + \frac{1}{2}\frac{(h + t)t}{a^2}\left(\frac{E_f}{E_c} + \frac{E_f}{G_c}\right) - \frac{D^*}{a^2C^*}\left(1 + \frac{\lambda}{3}\right)\right] \end{aligned}\right\} \qquad (152)$$

As further conditions, it is prescribed that $v(0) = v\left(\frac{\pi}{2}\right) = 0$, which makes $A_2 = A_1 = 0$ in equation (151). There remains

$$\left.\begin{aligned} v &= A\theta \cos \theta \\ w &= A\theta \sin \theta - B \cos \theta \end{aligned}\right\} \qquad (153)$$

Of particular interest are the values of $w(\pi/2)$ and $w(0)$, the first of these giving the radial deflection of the point of load application, the second giving the change of radius at right angles to the applied load. It is found that

$$w\left(\frac{\pi}{2}\right) = A\frac{\pi}{2} = \frac{\pi}{4}\frac{Q_0 a^3}{D^*}\left[1 + \frac{1}{2}\frac{(h + t)t}{a^2}\left(\frac{E_f}{E_c} + \frac{E_f}{G_c}\right) + \frac{D^*}{a^2C^*}\left(1 + \frac{\lambda}{3}\right)\right] \qquad (154)$$

$$w(0) = -B = -\frac{1}{2}\frac{Q_0 a^3}{D^*}\left[1 + \frac{1}{2}\frac{(h + t)t}{a^2}\left(\frac{E_f}{E_c} + \frac{E_f}{G_c}\right) - \frac{D^*}{a^2C^*}\left(1 + \frac{\lambda}{3}\right)\right] \qquad (155)$$

Equations (154) and (155) contain the interesting result that, for this problem, transverse shear and transverse normal stress affect the outcome formally in nearly the same way. If the generally unimportant terms with D^*/a^2C^* are omitted, which amounts to the usual assumption of circumferential inextensibility of the ring, then the effects of finite E_c and G_c occur in exactly the same way.

For a numerical example take $h = 0.9$ inch, $t = 0.05$ inch, $a = 20$ inches, $E_f/E_c = 1000$, and $E_f/G_c = 2000$. This gives

$$\frac{1}{2}\frac{(h + t)t}{a^2} = \frac{1}{16,800}$$

$$\frac{D^*}{a^2C^*} = \frac{1}{4}\frac{(h + t)^2}{a^2} = \frac{1}{1770}$$

$$\lambda = \frac{1}{2}\frac{(h + t)t}{a^2}\frac{E_f}{E_c} = \frac{1}{16.8}$$

$$\frac{1}{2}\frac{(h + t)t}{a^2}\frac{E_f}{G_c} = \frac{2}{16.8}$$

The factors in brackets in equations (154) and (155) become

$$1 + \frac{1}{16.8} + \frac{2}{16.8} + \frac{1}{1770}\left(1 + \frac{1}{3 \times 16.8}\right) = 1.18$$

and

$$1 + \frac{1}{16.8} + \frac{2}{16.8} - \frac{1}{1770}\left(1 + \frac{1}{3 \times 16.8}\right) = 1.18$$

Thus, in the present example the flexibility of the core is responsible for an 18-percent increase of deflection-load ratio, and of this 12 percent is due to transverse shearing and 6 percent to transverse normal stress. Compared with these two effects the effect of circumferential extensibility of the composite ring is seen to be negligible. As a further numerical illustration, it is noted that reducing the ring radius a from 20 inches to 10 inches, with all other data unchanged, changes the 18-percent correction to a 72-percent correction.

Bending of complete circular ring under action of two concentrated radial forces at $\theta = \pm\pi/2$.- The solution of this problem may be obtained by superposition of the solutions for the semicircular ring under the action of end shear forces Q_0 (equations (146) to (155)) and under the action of end loading moments M_0 (equations (126) to (129)).

The first step consists in determining M_0 in terms of Q_0 such that the sum of the β's from equations (129) and (149) assumes the value zero for $\theta = \pi/2$, that is, the value of the superimposed bending moment at $\theta = \pi/2$ must make the tangent to the deflected ring at this point horizontal. Combining equations (129) and (149) in this manner, there is obtained

$$\frac{\pi}{2}\frac{1 + \lambda}{D^*} aM_0 - \frac{1 + \lambda}{D^*} a^2 Q_0 = 0$$

or

$$M_0 = (2/\pi) a Q_0 \tag{156}$$

It may be noted that equation (156) is a further case of a statically indeterminate problem where transverse shear and normal stress flexibility do not affect the internal force and moment distribution but affect only the state of deformation of the structure.

Further, the radial deflections $w(\pi/2)$ and $w(0)$ due to the action of M_0 are calculated, in order to combine them with equations (152) and (153). Integrating equations (129) and (127) with the boundary conditions $v(0) = v(\pi/2) = 0$, there is obtained for the displacements due to M_0,

$$\left.\begin{aligned} D^*w &= -(1 + \lambda)M_0 a^2\left(1 - \frac{\pi}{2}\cos\theta\right) \\ D^*v &= (1 + \lambda)M_0 a^2\left(\theta - \frac{\pi}{2}\sin\theta\right) \end{aligned}\right\} \tag{157}$$

and, in particular,

$$\left.\begin{aligned} D^*w(0) &= (1 + \lambda)M_0 a^2\left(\frac{\pi}{2} - 1\right) \\ D^*w\left(\frac{\pi}{2}\right) &= -(1 + \lambda)M_0 a^2 \end{aligned}\right\} \tag{158}$$

Combining equations (158) with equations (154) and (155) and taking M_0 from equation (156), there follows for the resultant displacements

$$\left.\begin{aligned} w\left(\frac{\pi}{2}\right) &= \frac{Q_0a^3}{D^*}\left\{\left(\frac{\pi}{4} - \frac{2}{\pi}\right)(1 + \lambda) + \frac{\pi}{4}\left[\lambda_G + \frac{D^*}{a^2C^*}\left(1 + \frac{\lambda}{3}\right)\right]\right\} \\ w(0) &= -\frac{Q_0a^3}{D^*}\left\{\left(\frac{2}{\pi} - \frac{1}{2}\right)(1 + \lambda) + \frac{1}{2}\left[\lambda_G - \frac{D^*}{a^2C^*}\left(1 + \frac{\lambda}{3}\right)\right]\right\} \end{aligned}\right\} \tag{159}$$

where $\lambda_G = \frac{(h + t)t}{2a^2} \frac{E_f}{G_c}$ has been put as a further abbreviation. Equations (159) may be written in the alternate form

$$w(\pi/2) = 0.149 \frac{Q_0a^3}{D^*}\left\{1 + \lambda + 5.29\left[\lambda_G + \frac{D^*}{a^2C^*}\left(1 + \frac{\lambda}{3}\right)\right]\right\} \tag{160}$$

$$w(0) = -0.137 \frac{Q_0a^3}{D^*}\left\{1 + \lambda + 3.65\left[\lambda_G - \frac{D^*}{a^2C^*}\left(1 + \frac{\lambda}{3}\right)\right]\right\} \tag{161}$$

When $\lambda = \lambda_G = 0$ and when the composite ring is assumed axially inextensible, which amounts to putting $D^*/a^2C^* = 0$ in equations (160) and (161), then equations (160) and (161) reduce to well-known results of circular-ring analysis.

Comparing equations (160) and (161) for the closed circular ring with equations (154) and (155) for the open semicircular ring, it is noteworthy that for the semicircular ring λ and λ_G occur with equal weight, while for the closed circular ring the influence of λ_G is considerably greater than the influence of λ. Thus, for the closed circular ring the effect of transverse shear deformation is much more important than the effect of transverse normal stress deformation, while for the open semicircular ring both effects occur in a much more nearly equally important way.

For a numerical example of the use of equations (160) and (161) take again the values for the numerical example given in the section entitled "Bending of semicircular ring by end shear forces." This gives for the expressions in braces

$$1 + \frac{1}{16.8} + \frac{2 \times 5.29}{16.8} + \frac{5.29}{1770} = 1.69$$

and

$$1 + \frac{1}{16.8} + \frac{2 \times 3.65}{16.8} + \frac{3.65}{1770} = 1.50$$

Thus, while the effect of transverse stress deformation for the open circular ring amounted to 18 percent, the corresponding corrections for the closed ring are 69 and 50 percent, respectively.

The next step in the analysis of sandwich-type circular rings would be the general solution of the system of equations (120) to (124) for arbitrary load distributions. This, evidently, is possible and further specific examples of interest might be analyzed on the basis of the general solution. Such extension of the work of this section is, however, left for future considerations.

Circular Cylindrical Shells

In this section the general system of equations of part I of this report is restricted to the equations of the theory of circular cylindrical shells. The treatment of sandwich-type shells of this kind is shown to be not appreciably more difficult than the analysis without the effect of transverse shear and normal stress.

As specific examples some problems of rotationally symmetric deformations are treated. In particular the influence coefficients are obtained for a semi-infinite shell acted upon by bending moments and transverse forces at one end of the semi-infinite shell. With these influence coefficients an explicit solution is obtained for the problem of the infinite circular cylindrical shell acted upon by a pressure band of zero width.

In the general equations of the problem there are set for the relevant coordinates and variables,

$$\left.\begin{array}{lll}
\xi_1 = a\theta & \xi_2 = x & R_1 = a \\
R_2 = \infty & N_{11} = N_\theta & N_{22} = N_x \\
Q_1 = Q_\theta & Q_2 = Q_x & M_{11} = M_\theta \\
M_{22} = M_x & u_1 = v & u_2 = u \\
\beta_1 = \beta_\theta & \beta_2 = \beta_x & m_1 = m_\theta \\
m_2 = m_x & p_1 = p_\theta & p_2 = p_x \\
\multicolumn{3}{c}{\alpha_1 = \alpha_2 = 1} \\
\multicolumn{3}{c}{N_{12} = N_{21} = N_{x\theta}} \\
\multicolumn{3}{c}{M_{12} = M_{21} = M_{x\theta}}
\end{array}\right\} \tag{162}$$

The equilibrium differential equations (25) to (28), (31), and (34) become

$$\left.\begin{aligned}
\frac{\partial N_x}{\partial x} + \frac{1}{a}\frac{\partial N_{x\theta}}{\partial \theta} + p_x &= 0 \\
\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{a}\frac{\partial N_\theta}{\partial \theta} + \frac{Q_\theta}{a} + p_\theta &= 0 \\
\frac{\partial Q_x}{\partial x} + \frac{1}{a}\frac{\partial Q_\theta}{\partial \theta} - \frac{N_\theta}{a} + q &= 0
\end{aligned}\right\} \tag{163}$$

$$\left.\begin{aligned}
\frac{\partial M_x}{\partial x} + \frac{1}{a}\frac{\partial M_{x\theta}}{\partial \theta} - Q_x + m_x &= 0 \\
\frac{\partial M_{x\theta}}{\partial x} + \frac{1}{a}\frac{\partial M_\theta}{\partial \theta} - Q_\theta + m_\theta &= 0
\end{aligned}\right\} \tag{164}$$

$$\sigma_{\zeta_m} = s - \frac{M_\theta}{(h + t)a} \tag{165}$$

The stress-strain relations (56), (59), (60), (61), and (64) to (67) become, with $\lambda_2 = \lambda_{12} = 0$, $\lambda_1 = \frac{1}{2}\left[(h + t)t/a^2\right](E_f/E_c) \equiv \lambda$,

$$\left.\begin{aligned}
\left(1 + \frac{\lambda}{3}\right)N_\theta - \nu N_x &= C^*\left[\frac{1}{a}\frac{\partial v}{\partial \theta} + \frac{w}{a} + \frac{(h + t)q}{12aE_c}\right] \\
N_x - \nu N_\theta &= C^*\left(\frac{\partial u}{\partial x}\right) \\
2(1 + \nu)N_{x\theta} &= C^*\left(\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta}\right)
\end{aligned}\right\} \tag{166}$$

$$\left.\begin{aligned}
Q_\theta &= (h + t)G_c\left(\beta_\theta + \frac{1}{a}\frac{\partial w}{\partial \theta} - \frac{v}{a}\right) \\
Q_x &= (h + t)G_c\left(\beta_x + \frac{\partial w}{\partial x}\right)
\end{aligned}\right\} \tag{167}$$

$$\left.\begin{aligned}
(1 + \lambda)M_\theta - \nu M_x &= D^*\left(\frac{1}{a}\frac{\partial \beta_\theta}{\partial \theta} + \frac{s}{aE_c}\right) \\
M_x - \nu M_\theta &= D^*\left(\frac{\partial \beta_x}{\partial x}\right) \\
2(1 + \nu)M_{x\theta} &= D^*\left(\frac{\partial \beta_\theta}{\partial x} + \frac{1}{a}\frac{\partial \beta_x}{\partial \theta}\right)
\end{aligned}\right\} \tag{168}$$

When $G_c = E_c = \infty$ (and therewith $\lambda = 0$) equations (163), (164), (166), (167), and (168) reduce to the known system of equations in which deformations due to transverse stresses are neglected. The solution of the present system of equations is not essentially more difficult than the solution of the system with $G_c = E_c = \infty$. In particular also here

there may be obtained a trigonometric double-series solution, as a generalization of Navier's solution for the flat plate (references 7 and 8).

For this trigonometric double-series solution there is set,

$$\left.\begin{aligned}
q &= \sum\sum q_{mn} \sin m\theta \sin nx/l \\
p_\theta &= \sum\sum p_{\theta mn} \cos m\theta \sin nx/l \\
p_x &= \sum\sum p_{xmn} \sin m\theta \cos nx/l \\
m_x &= \sum\sum m_{xmn} \sin m\theta \cos nx/l \\
m_\theta &= \sum\sum m_{\theta mn} \cos m\theta \sin nx/l \\
s &= \sum\sum s_{mn} \sin m\theta \sin nx/l
\end{aligned}\right\} \qquad (169)$$

$$\left.\begin{aligned}
w &= \sum\sum w_{mn} \sin m\theta \sin nx/l \\
v &= \sum\sum v_{mn} \cos m\theta \sin nx/l \\
u &= \sum\sum u_{mn} \sin m\theta \cos nx/l \\
\beta_x &= \sum\sum \beta_{xmn} \sin m\theta \cos nx/l \\
\beta_\theta &= \sum\sum \beta_{\theta mn} \cos m\theta \sin nx/l
\end{aligned}\right\} \qquad (170)$$

$$
\left.
\begin{aligned}
Q_x &= \sum \sum Q_{xmn} \sin m\theta \cos nx/l \\
Q_\theta &= \sum \sum Q_{\theta mn} \cos m\theta \sin nx/l \\
\left(N_x, N_\theta\right) &= \sum \sum \left(N_{xmn}, N_{\theta mn}\right) \sin m\theta \sin nx/l \\
N_{x\theta} &= \sum \sum N_{x\theta mn} \cos m\theta \cos nx/l \\
\left(M_x, M_\theta\right) &= \sum \sum \left(M_{xmn}, M_{\theta mn}\right) \sin m\theta \sin nx/l \\
M_{x\theta} &= \sum \sum M_{x\theta mn} \cos m\theta \cos nx/l
\end{aligned}
\right\} \qquad (171)
$$

When equations (169) to (171) are substituted in equations (163) to (168) there remains for every value of m and n a system of 13 simultaneous equations for the 13 Fourier coefficients which occur in equations (170) and (171).

A system of only five simultaneous equations for the five Fourier coefficients in equation (170) is obtained if first equations (163) and (164) are reduced to five equations for the five unknowns w, v, u, β_x, and β_θ, by means of equations (166) to (168).

For the present, the task is not carried out of obtaining the deformation and internal stress Fourier coefficients of equations (170) and (171) in terms of the Fourier coefficients of the load terms in equation (169). Instead, the axisymmetrical case, to which equations (169) to (171) reduce when $\sin m\theta$ and $\cos m\theta$ are interchanged throughout, and then only the terms for $m = 0$ are taken, is treated separately.

<u>Axisymmetrical deformation of circular cylindrical shell</u>.- In equations (163) to (168) set

$$
\left.
\begin{aligned}
&\partial(\ )/\partial\theta = 0 \\
&\partial(\ )/\partial x = (\ )' \\
&N_{x\theta} = Q_\theta = M_{\theta x} = 0 \\
&v = \beta_\theta = 0 \\
&m_\theta = p_\theta = 0
\end{aligned}
\right\} \qquad (172)
$$

and then the following system of equations has to be dealt with:

$$\left.\begin{aligned} & N_x' + p_x = 0 \\ & Q_x' - (N_\theta/a) + q = 0 \end{aligned}\right\} \tag{173}$$

$$\left.\begin{aligned} & M_x' - Q_x + m_x = 0 \\ & \sigma\zeta_m = s - M_\theta/(h + t)a \end{aligned}\right\} \tag{174}$$

$$\left.\begin{aligned} & \left(1 + \frac{\lambda}{3}\right)N_\theta - \nu N_x = C^*\left[\frac{w}{a} + \frac{(h + t)q}{12aE_c}\right] \\ & N_x - \nu N_\theta = C^*u' \end{aligned}\right\} \tag{175}$$

$$Q_x = (h + t)G_c(\beta_x + w') \tag{176}$$

$$\left.\begin{aligned} & (1 + \lambda)M_\theta - \nu M_x = D^*s/aE_c \\ & M_x - \nu M_\theta = D^*\beta_x' \end{aligned}\right\} \tag{177}$$

The system of equations (173) to (177) may be reduced to two simultaneous equations for β_x and Q_x, as follows: First, express M_x in terms of β_x by means of equation (175) and substitute the result in equation (174). From the first of equations (177), it follows that

$$M_\theta = \frac{\nu}{1 + \lambda} M_x + \frac{D^*s}{(1 + \lambda)aE_c} \tag{178}$$

and this, introduced into the second of equations (177), gives

$$M_x = \frac{(1 + \lambda)D^*}{1 + \lambda - \nu^2} \beta_x' + \frac{\nu D^* s}{aE_c(1 + \lambda - \nu^2)} \tag{179}$$

Equation (179) is introduced into the first of equations (174) and, restricting attention to shells of uniform section properties, there is obtained

$$\frac{(1 + \lambda)D^*}{1 + \lambda - \nu^2} \beta_x'' - Q_x = -m_x - \frac{\nu D^* s}{aE_c(1 + \lambda - \nu^2)} \tag{180)*$$

To obtain the second of these equations, first, introduce into equation (176) the value of w' which follows from equation (175), giving

$$\frac{Q_x}{(h + t)G_c} = \beta_x + \frac{a}{C^*}\left[\left(1 + \frac{\lambda}{3}\right)N_\theta' - \nu N_x' - \frac{C^*(h + t)q'}{12aE_c}\right] \tag{181}$$

In equation (181), N_θ' and N_x' are taken from equation (173) and, after slight transformations, there is obtained

$$\frac{a^2}{C^*}\left(1 + \frac{\lambda}{3}\right)Q_x'' - \frac{Q_x}{(h + t)G_c} + \beta_x = -\frac{a^2}{C^*}\left(q' + \frac{\nu p_x}{a}\right) \tag{182)*$$

Comparing equations (180) and (182) with the corresponding equations without the effect of transverse shear and normal stress deformation, it is seen that the effect of transverse normal stress, which is represented by λ, merely somewhat modifies some of the coefficients of the left sides of the corresponding system of equations with $E_c = \infty$. In contrast to this, the effect of finite G_c is to introduce a new term into the left sides of these equations. This new term may be of appreciable importance, as will be shown.

Having solved equations (180) and (182), M_x and M_θ are obtained from equations (179) and (178), respectively; N_θ follows from equation (173) in the form

$$N_\theta = a(Q_x' + q) \tag{183}$$

and w follows from equation (175) in the form

$$w = (a/C^*)\left[(1 + \lambda/3)aQ_x' + aq + \nu\int p_x \, dx\right] \tag{184}$$

The following examples illustrate the use of equations (178) to (184).

<u>Infinite circular cylindrical shell with periodic load distribution</u>.– In specialization of equations (169) to (171), set

$$\left.\begin{aligned} q &= q_\mu \sin \mu x \\ s &= s_\mu \sin \mu x \\ p_x &= p_{x\mu} \cos \mu x \\ m_x &= m_{x\mu} \cos \mu x \end{aligned}\right\} \tag{185}$$

$$\left.\begin{aligned} w &= w_\mu \sin \mu x \\ u &= u_\mu \cos \mu x \\ \beta_x &= \beta_{x\mu} \cos \mu x \end{aligned}\right\} \tag{186}$$

$$\left.\begin{aligned} Q_x &= Q_{x\mu} \cos \mu x \\ N_x &= N_{x\mu} \sin \mu x \\ N_\theta &= N_{\theta\mu} \sin \mu x \\ M_x &= M_{x\mu} \sin \mu x \\ M_\theta &= M_{\theta\mu} \sin \mu x \end{aligned}\right\} \tag{187}$$

By introducing equations (185) to (187) into equations (180) and (182), two simultaneous equations are obtained for the amplitudes $Q_{x\mu}$ and $\beta_{x\mu}$, as follows:

$$\left.\begin{aligned} &\frac{(1+\lambda)D^*}{1+\lambda-\nu^2}\mu^2\beta_{x\mu} + Q_{x\mu} = m_{x\mu} + \frac{\nu D^*}{1+\lambda-\nu^2}\frac{\mu s_\mu}{aE_c} \\ &\beta_{x\mu} - \frac{a^2}{C^*}\left[\left(1+\frac{\lambda}{3}\right)\mu^2 + \frac{2tE_f}{a^2(h+t)G_c}\right]Q_{x\mu} = -\frac{a^2}{C^*}\left(\mu q_\mu + \nu p_{x\mu}/a\right) \end{aligned}\right\} \quad (188)$$

To simplify the further discussion, by setting in equation (188) $m_{x\mu} = s_\mu = p_{x\mu} = 0$, there is obtained for $\beta_{x\mu}$ and $Q_{x\mu}$

$$\left.\begin{aligned} \beta_{x\mu} &= \frac{1+\lambda-\nu^2}{(1+\lambda)D^*}\frac{q_\mu}{\mu^3}K \\ Q_{x\mu} &= \frac{q_\mu}{\mu}K \end{aligned}\right\} \quad (189)$$

The quantity K is given by

$$K = \left[1 + \frac{\lambda}{3} + \frac{2}{\pi^2}\frac{l^2t}{(h+t)a^2}\frac{E_f}{G_c} + \frac{1+\lambda-\nu^2}{(1+\lambda)\pi^4}\frac{4l^4}{(h+t)^2a^2}\right]^{-1} \quad (190)$$

where use has been made of the relation $\mu = \pi/l$. In equation (190) the term $\lambda/3$ will usually be of little importance. The other two variable terms represent the effect of transverse shear deformation and of shell curvature, respectively. When the radius a is so large that $l^4/(h+t)^2a^2 \ll 1$, the shell behaves under the action of the given load essentially as a plate strip. The effect of transverse shear is important as soon as the term $(2/\pi^2)(l^2/a^2)(t/(h+t))(E_f/G_c)$ is not small compared with 1.

Before evaluating a numerical example the following further formulas which are readily obtained from equations (179), (183), and (184) are listed:

$$\left.\begin{aligned} M_{x\mu} &= \frac{q_\mu}{\mu^2} K \\ N_{\theta\mu} &= a q_\mu (1 - K) \\ w_\mu &= \frac{a^2 q_\mu}{C^*}\left[1 - \left(1 + \frac{\lambda}{3}\right)K\right] \end{aligned}\right\} \tag{191}$$

Equations (191) show that in this problem not only is the deflection increased because of the effect of transverse shear, and with that the hoop stress resultant $N_{\theta\mu}$, but now also an effect is found on the bending-moment distribution $M_{x\mu}$, in the opposite sense. The effect of transverse shear is to reduce the magnitude of the bending moments in the shell. This result is in contrast to what was found for the examples which were worked out in the sections on plate analysis and circular ring analysis and is therefore of particular significance.

Equation (191) for w_μ may be compared with the corresponding expression for a simply supported plate strip of width l, with sinusoidal load. The result for this case must follow from equation (191) in the limit $a \rightarrow \infty$ and agree with equation (96), which was previously obtained. To compare the last of equations (191) with equation (96), the last of equations (191) is written in the form

$$w_\mu = \frac{\dfrac{1 + \lambda - \nu^2}{(1 + \lambda)D^*}\left(\dfrac{l}{\pi}\right)^4}{1 + \dfrac{1 + \lambda - \nu^2}{(1 + \lambda)D^*}\left(\dfrac{l}{\pi}\right)^4 \dfrac{C^*}{a^2}} \times \frac{1 + \left(\dfrac{\pi}{l}\right)^2 \dfrac{1 + \lambda}{1 + \lambda - \nu^2} \dfrac{1}{2} \dfrac{(h + t)t}{l^2} \dfrac{E_f}{G_c}}{1 + \dfrac{\dfrac{\lambda}{3} + \left(\dfrac{l}{\pi}\right)^2 \dfrac{2t}{h + t} \dfrac{E_f}{G_c}}{1 + \dfrac{1 + \lambda - \nu^2}{(1 + \lambda)D^*}\left(\dfrac{l}{\pi}\right)^4 \dfrac{C^*}{a^2}}} q_\mu \tag{192}$$

Equation (192) reduces to the equivalent of equation (96) if in it $a \rightarrow \infty$.

From a comparison of equations (192) and (96), it is further concluded that the correction due to transverse shear is greatest in this case when $a = \infty$, so that, in this case, the curvature of the shell tends to reduce the additional shear deformation, below the value obtained for the simply supported plate strip.

For a numerical example first take $h = 1$ inch, $t = 0.05$ inch, $a = 10$ inches, $l = 20$ inches, $E_f/G_c = 200$, $E_f/E_c = 100$, $\nu = 1/3$, and $\lambda = \frac{1}{2}\,\frac{1.05 \times 0.05}{100}\,100 = 0.025$. The factor K of equation (190) becomes

$$K = \left[1 + 0.008 + \frac{2}{\pi^2}\,\frac{400 \times 0.05}{100 \times 1.05} \times 100 + \frac{4(1 - 0.09)}{\pi^4}\,\frac{160,000}{100 \times 1.1}\right]^{-1}$$

$$= (1 + 0.008 + 3.86 + 54.5)^{-1} = 0.01685$$

while without transverse shear and normal stress deformation

$$(K)_{G_c=E_c=\infty} = (1 + 54.5)^{-1} = 0.0180$$

The correction in this case amounts to about 6 percent.

Changing the moduli ratio to $E_f/G_c = 2000$, $E_f/E_c = 1000$,

$$K = (1 + 0.08 + 38.6 + 54.5) = 0.0106$$

instead of $K = 0.01685$. The correction in this case amounts to $\frac{0.0180 - 0.0106}{0.0106} \times 100 \approx 70$ percent. Thus again there is a case where omission of the effect of transverse shear deformation would give results which could not be used. However, it is noted that the effect of transverse normal stress deformation is quite small and may here safely be neglected.

If the foregoing values of K are introduced into equations (191), it is seen that the percentage corrections apply to the bending-moment value directly but that for hoop tension and radial deflection the corrections are very small indeed. In fact, in order that there be appreciable corrections due to transverse shear on hoop tension and radial deflection, it is necessary that the half wave length of the sinusoidal load q be so small that K is at least of magnitude 0.25 or more.

A case of approximately this kind is obtained if the half wave length l is changed from 20 inches to 10 inches and the moduli ratios are again taken as $E_f/E_c = 100$, $E_f/G_c = 200$. Then,

$$K = (1 + 0.008 + 0.965 + 3.41)^{-1} = 0.1865$$

whereas

$$(K)_{E_c=G_c=\infty} = (1 + 3.41)^{-1} = 0.227$$

The percentage change of K and therewith of M_x is slightly more than 19. The percentage change of N_θ and w is about $4\frac{1}{2}$.

The foregoing numerical examples show that the effect of transverse shear may be significant in cylindrical sandwich-shell analysis and that moreover its magnitude will not in general be predictable by the analysis of an equivalent flat-plate or straight-beam problem.

For the infinite circular cylindrical shell with load $q = q_\mu \cos \mu x$ the essential results are given by equations (190) and (191). These results may be extended directly to the loading condition

$$\left.\begin{aligned} q &= \sum q_n \cos \mu_n x \\ \mu_n &= n\pi/l \end{aligned}\right\} \qquad (193)$$

By superposition, from equation (191) the following formulas are obtained:

$$\left.\begin{aligned} M_x &= \sum \left(q_n/\mu_n^2\right)K_n \cos \mu_n x \\ N_\theta &= a \sum q_n(1 - K_n) \cos \mu_n x \\ w &= (a^2/C^*) \sum q_n\left[1 - (1 + \lambda/3)K_n\right] \cos \mu_n x \end{aligned}\right\} \qquad (194)$$

The values of K_n are obtained from the formula

$$K_n = \left[1 + \frac{\lambda}{3} + \frac{2}{n^2\pi^2} \frac{l^2 t}{(h + t)a^2} \frac{E_f}{G_c} + \frac{4(1 + \lambda - \nu^2)}{(1 + \lambda)\pi^4 n^4} \frac{l^4}{(h + t)^2 a^2}\right]^{-1} \tag{195}$$

Having the solution for the infinite shell with periodic load distribution, it will be only necessary to add to this the general solution of the differential equations without external load terms, in order to obtain the complete solution for any edge condition of the axisymmetrically stressed circular cylindrical shell of finite length. This additional solution will now be obtained.

Finite circular cylindrical shell acted upon by edge moments and forces.– To solve equations (180) and (182) with right-hand sides equal to zero, equation (182) is differentiated twice and β_x'' is substituted from equation (180). This gives

$$\frac{a^2}{C^*}\left(1 + \frac{\lambda}{3}\right)Q_x^{IV} - \frac{1}{(h + t)G_c} Q_x'' + \frac{1 + \lambda - \nu^2}{(1 + \lambda)D^*} Q_x = 0$$

or

$$Q_x^{IV} - 2m_1^2 Q_x'' + 4m_2^4 Q_x = 0 \tag{196}$$

where

$$m_1 = \frac{1}{a}\sqrt{\frac{\frac{1}{2} C^*}{(1 + \lambda/3)(h + t)G_c}} = \frac{1}{a}\sqrt{\frac{t}{(1 + \lambda/3)(h + t)} \frac{E_f}{G_c}} \tag{197a}$$

$$m_2 = \sqrt[4]{\frac{\frac{1}{4} C^*}{a^2 D^*} \frac{1 + \lambda - \nu^2}{(1 + \lambda)(1 + \lambda/3)}} = \frac{1}{\sqrt{(h + t)a}}\sqrt[4]{\frac{1 + \lambda - \nu^2}{(1 + \lambda)(1 + \lambda/3)}} \tag{197b}$$

The auxiliary equation corresponding to equation (196) is

$$r^4 - 2m_1^2r^2 + 4m_2^4 = 0 \tag{198a}$$

or

$$r^2 = m_1^2 \pm \sqrt{m_1^4 - 4m_2^4} \tag{198b}$$

The solution of equation (196) occurs in two different forms, depending on whether r^2 of equation (198b) is real or not. According to equations (197) and (198b), r^2 is complex as long as

$$\left.\begin{aligned} & m_1^4 < 4m_2^4 \\ & \text{or} \\ & \frac{1}{a^4}\left[\frac{tE_f}{(h + t)(1 + \lambda/3)G_c}\right]^2 < \frac{4\left[1 - \nu^2/(1 + \lambda)\right]}{(h + t)^2a^2(1 + \lambda/3)} \end{aligned}\right\} \tag{199a}$$

To clarify this condition, neglect λ (which is of very little importance here) and equation (199a) then becomes

$$\frac{E_f}{G_c} < \frac{2a}{t} \tag{199b}$$

When equation (199) holds, a quantity k may be defined by

$$k = \sqrt{m_1^2 + i\sqrt{4m_2^2 - m_1^4}} \tag{200}$$

and the four roots of the characteristic equation are k, $\overline{k}$, $-k$, and $-\overline{k}$, where a bar indicates the taking of conjugates. The solution of equation (196) may be written

$$Q_x = C_1e^{-kx} + \overline{C}_1e^{-\overline{k}x} + C_2e^{kx} + \overline{C}_2e^{\overline{k}x} \tag{201}$$

Where equation (199) does not hold, which is the case for very small values of G_c/E_f only, all four roots of equation (198a) are real and of the form

$$\left.\begin{aligned} k_1 &= \sqrt{m_1^2 + \sqrt{m_1^4 - 4m_2^2}} \\ k_2 &= -k_1 \\ k_3 &= \sqrt{m_1^2 - \sqrt{m_1^4 - 4m_2^2}} \\ k_4 &= -k_3 \end{aligned}\right\} \tag{202}$$

and the solution of equation (196) can be taken in the form

$$Q_x = A_1 e^{k_1 x} + A_2 e^{-k_1 x} + A_3 e^{k_3 x} + A_4 e^{-k_3 x} \tag{203}$$

Before applying either solution to a specific problem, there are noted the following relations which follow from equation (200)

$$\left.\begin{aligned} k\bar{k} &= |k|^2 = 2m_2^2 \\ k + \bar{k} &= \sqrt{2}\sqrt{m_1^2 + 2m_2^2} \end{aligned}\right\} \tag{204}$$

<u>Semi-infinite shell acted upon by edge bending moment and shear force</u>.— There are the following boundary conditions,

$$\left.\begin{aligned} M_x(0) &= \frac{(1 + \lambda)D^*}{1 + \lambda - \nu^2}\beta_x'(0) = M_0 \\ Q_x(0) &= Q_0 \end{aligned}\right\} \tag{205}$$

while for $x = \infty$ these same quantities vanish.[15]

---

[15]For the same problem without the effect of transverse shear and normal stress, see reference 9.

Of particular interest in this solution are the values of deflection $w(0)$ and change of slope $\beta_x(0)$ at the section where the loads M_0 and Q_0 are applied.[16]

Taking first the case $E_f/G_c < 2a/t$ for which equation (201) applies, it is seen that the conditions at infinity require that

$$C_2 = \overline{C}_2 = 0 \tag{206}$$

so that

$$Q_x = C_1 e^{-kx} + \overline{C}_1 e^{-\overline{k}x} \tag{207}$$

The values of β_x may be obtained by integration from equation (180) in the form

$$\frac{(1 + \lambda)D^*}{1 + \lambda - \nu^2} \beta_x = \frac{C_1}{k^2} e^{-kx} + \frac{\overline{C}_1}{\overline{k}^2} e^{-\overline{k}x} \tag{208}$$

where two constants of integration have been discarded to satisfy again the conditions at infinity.

With equations (207) and (208) there is obtained from the boundary conditions (equations (205)) that

$$\left.\begin{aligned} C_1 + \overline{C}_1 &= Q_0 \\ \left(C_1/k\right) + \left(\overline{C}_1/\overline{k}\right) &= -M_0 \end{aligned}\right\} \tag{209}$$

---

[16]Without transverse shear and normal stress deformation these relations are

$$w(0) = \sqrt{\frac{1 - \nu^2}{D^*C^*}}\, a \left[M_0 + \sqrt[4]{\frac{4a^2D^*}{(1 - \nu^2)C^*}}\, Q_0\right]$$

$$\beta_x(0) = \sqrt{\frac{1 - \nu^2}{D^*C^*}}\, a \left[Q_0 + 2\sqrt[4]{\frac{(1 - \nu^2)C^*}{4a^2D^*}}\, M_0\right]$$

in agreement with equations (236) of reference 9, where the homogeneous shell is considered.

This determines C_1 and $\overline{C}_1$ in the form

$$\left.\begin{aligned} C_1 &= \frac{kQ_0 + |k|^2 M_0}{k - \overline{k}} \\ \overline{C}_1 &= \frac{\overline{k}Q_0 + |k|^2 M_0}{\overline{k} - k} \end{aligned}\right\} \tag{210}$$

Equation (210) is introduced into equation (208) and there is obtained as the first of two "influence coefficient" formulas

$$\frac{(1 + \lambda)D^*}{1 + \lambda - \nu^2}\beta_x(0) = -\frac{1}{|k|^2}\left[Q_0 + (k + \overline{k})M_0\right] \tag{211}$$

The second of these formulas follows from equations (184), (207), and (210) in the form

$$\frac{C^*}{1 + \lambda/3}\frac{w(0)}{a^2} = -\left[|k|^2 M_0 + (k + \overline{k})Q_0\right] \tag{212}$$

Equations (211) and (212) may be written in more explicit form, using equations (204) and (197). The results are

$$\beta_x(0) = -\frac{a}{\sqrt{C^*D^*}}\sqrt{\frac{(1 + \lambda/3)(1 + \lambda - \nu^2)}{1 + \lambda}}\left[Q_0 \right.$$

$$\left. + \sqrt{\sqrt{\frac{4C^*}{a^2D^*}\frac{1 + \lambda - \nu^2}{(1 + \lambda)(1 + \lambda/3)}} + \frac{(C^*/a^2G_c)}{(h + t)(1 + \lambda/3)}}\; M_0\right] \tag{213}$$

and

$$w(0) = -\frac{a}{\sqrt{C^*D^*}}\sqrt{\frac{(1 + \lambda/3)(1 + \lambda - \nu^2)}{1 + \lambda}}\,M_0$$

$$-\frac{a^2}{C^*}\left(1 + \frac{\lambda}{3}\right)\sqrt{\sqrt{\frac{4C^*}{a^2D^*}\,\frac{1 + \lambda - \nu^2}{(1 + \lambda)(1 + \lambda/3)}} + \frac{(C^*/a^2G_c)}{(h + t)(1 + \lambda/3)}}\,Q_0 \quad (214)$$

Neglecting the generally small effect of finite E_c in equations (213) and (214), that is, putting $\lambda = 0$ in these equations, there may be written instead

$$\beta_x(0) = -\frac{a\sqrt{1 - \nu^2}}{\sqrt{C^*D^*}}\left[Q_0 + \sqrt[4]{\frac{4(1 - \nu^2)\,C^*}{a^2D^*}}\sqrt{1 + \frac{t/a}{2\sqrt{1 - \nu^2}}\,\frac{E_f}{G_c}}\,M_0\right] \qquad (215)^*$$

$$w(0) = -\frac{a\sqrt{1 - \nu^2}}{\sqrt{C^*D^*}}\,M_0 - \frac{a^2}{C^*}\sqrt[4]{\frac{4(1 - \nu^2)\,C^*}{a^2D^*}}\sqrt{1 + \frac{t/a}{2\sqrt{1 - \nu^2}}\,\frac{E_f}{G_c}}\,Q_0 \qquad (216)^*$$

Equations (215) and (216) contain the noteworthy fact that the correction factors for the effect of transverse shear are independent of the ratio t/h of face-layer thickness to core thickness. The complete formulas of course must and do contain the influence of the core thickness h.

It is further noted that, while equations (211) to (216) have been derived for the case that $m_1^4 < 4m_2^4$, for which the complex solution holds, they are also valid, as is readily shown, when $4m_2^4 \geqslant m_1^4$.

Comparing equations (213) and (214), and (215) and (216) with the equations listed in footnote 16 it is seen that: (1) The effect of transverse shear modifies the deflection due to Q_0 and the rotation due to M_0 but not the other two coefficients, (2) the effect of transverse normal stress enters all four coefficients but only in a minor way, and (3) the reciprocity relation that the deflection due to M_0 is the same as the rotation due to Q_0 is carried over from the theory without the extra effects.

For a numerical example the following data are chosen: $t = 0.1$ inch, $h = 1$ inch, $a = 10$ inches, $E_f/E_c = 100$, $E_f/G_c = 200$, $\nu = 1/3$, and $\lambda = \frac{1}{2}\frac{1.1 \times 0.1}{100} 100 = 0.055$, and, from equation (197),

$$m_1 = \frac{1}{10}\sqrt{\frac{0.1 \times 200}{1.018 \times 1.1}} = 0.426$$

$$m_2 = \sqrt[4]{\frac{1 - 0.09}{100 \times 1.21 \times 1.018}} = 0.294$$

Then, according to equation (204),

$$|k|^2 = 0.173$$

$$k + \bar{k} = \sqrt{2}\sqrt{0.182 + 0.173} = 0.84$$

while without transverse shear deformation $(m_1 = 0)$ the value of $k + \bar{k} = 0.59$. According to equations (211) and (212), the effect of transverse shear in this case is to increase the rotation due to the edge moment in the ratio $0.84/0.59 = 1.42$, an effect of 42 percent. The same increase is found for the deflection due to the edge shear force. Rotation due to the shear force and deflection due to the moments are practically unchanged. Likewise, the effect of transverse normal stress in this case is of negligible importance.

As a further numerical example there is chosen $t = 0.05$ inch, $h = 1$ inch, $a = 20$ inches, $E_f/E_c = 1000$, $E_f/G_c = 2000$, and $\lambda = \frac{1}{2}\frac{1.05 \times 0.05}{400} 1000 = 0.065$, and, from equation (197),

$$m_1 = \frac{1}{20}\sqrt{\frac{0.05}{1.022 \times 1.05} 2000} = 0.483$$

$$m_2 = \frac{1}{\sqrt{1.05 \times 20}}\sqrt[4]{\frac{1 - 0.09}{1.022}} = 0.218$$

From equation (204) then

$$|k|^2 = 0.095$$

$$k + \overline{k} = \sqrt{2}\sqrt{0.234 + 0.095} = 0.82$$

while without transverse shear deformation $(m_1 = 0)$ the value of $k + \overline{k} = 0.44$. Thus the effect in this case is to increase edge rotation due to edge moment and edge deflection due to edge shear force in the ratio $0.82/0.44 = 1.87$, an effect of 87 percent.

Infinite circular cylindrical shell acted upon by transverse line load.— Calculation is restricted to the determination of deflection and bending moment at the section $x = 0$ where the line load of intensity $2Q_0$ is assumed to act. The result of the foregoing paragraph may be used as follows. Consider the infinite shell cut in two parts at the section $x = 0$ and assume a bending moment M_0 of such magnitude that the slope $\beta_x(0)$ is zero. According to equation (211), this gives

$$M_0 = -\frac{Q_0}{k + \overline{k}} = -\frac{Q_0}{\sqrt{2m_1^2 + 4m_2^2}} \tag{217}$$

and therewith

$$\frac{C^*w(0)}{(1 + \lambda/3)a^2} = \frac{(k + \overline{k})^2 - |k|^2}{k + \overline{k}} Q_0 = -\frac{2m_1^2 + 2m_2^2}{\sqrt{2m_1^2 + 4m_2^2}} Q_0 \tag{218}$$

Equations (217) and (218) become, with equations (204) and (197),

$$M_0 = \frac{Q_0}{\sqrt{2\sqrt{\frac{C^*}{a^2D^*}\frac{1 + \lambda - \nu^2}{(1 + \lambda)(1 + \lambda/3)}} + \frac{C^*/a^2}{(h + t)G_c}\frac{1}{1 + \lambda/3}}} \tag{219}$$

$$w(0) = -\frac{\left(1 + \frac{\lambda}{3}\right)a^2}{C^*} \frac{\sqrt{\frac{C^*}{a^2D^*} \frac{1 + \lambda - \nu^2}{(1 + \lambda)(1 + \lambda/3)} + \frac{C^*/a^2}{(h + t)G_c} \frac{1}{1 + \lambda/3}}}{\sqrt{2\sqrt{\frac{C^*}{a^2D^*} \frac{1 + \lambda - \nu^2}{(1 + \lambda)(1 + \lambda/3)}} + \frac{C^*/a^2}{(h + t)G_c} \frac{1}{1 + \lambda/3}}} \tag{220}$$

To give these formulas a somewhat less unwieldy appearance, the effect of finite E_c, that is, $\lambda = 0$, may again be neglected, as is permissible in most cases; and there may be written

$$M_0 = \frac{\sqrt{(h + t)a}}{2\sqrt[4]{(1 - \nu^2)}} \frac{-Q_0}{\sqrt{1 + \frac{1}{2\sqrt{1 - \nu^2}} \frac{t}{a} \frac{E_f}{G_c}}} \tag{221*}$$

$$w(0) = -\frac{\sqrt{1 - \nu^2}}{4} \frac{a^2}{t\sqrt{(h + t)a}} \frac{1 + \frac{1}{\sqrt{1 - \nu^2}} \frac{t}{a} \frac{E_f}{G_c}}{\sqrt{1 + \frac{1}{2\sqrt{1 - \nu^2}} \frac{t}{a} \frac{E_f}{G_c}}} \frac{Q_0}{E_f} \tag{222*}$$

Some numerical examples are given as follows.

Taking $t = 0.1$ inch, $a = 10$ inches, $E_f/G_c = 200$, and $\nu = 1/3$, transverse shear deformation reduces M_0 to $1/\sqrt{2.05}$ times the value which holds when $G_c = \infty$, that is, there is about a 30 percent reduction in M_0. At the same time the deflection under the line load is $3.05/\sqrt{2.05} = 2.14$ times what it is when $G_c = \infty$; that is, there is an increase of about 115 percent in $w(0)$.

Taking $t = 0.05$ inch, $a = 20$ inches, and $E_f/G_c = 200$, M_0 is decreased by a factor $\sqrt{4/5} = 0.89$, while $w(0)$ is increased by a factor $1.5/\sqrt{1.25} = 1.34$.

Taking $t = 0.05$ inch, $a = 20$ inches, and $E_f/G_c = 2000$, M_0 is decreased by a factor $1/\sqrt{3.62} = 0.526$, while $w(0)$ is increased by a factor $6.25/\sqrt{3.62} = 3.29$.

Equation (220) for $w(0)$ may be compared with equation (116) for the circular plate of radius a. This comparison shows that, while for the plate both the ratios t/a and $(h + t)/a$ enter into the correction factor, the correction factor for the cylindrical shell contains the ratio t/a only; that is, the corrections (but not the results) are independent of the ratio of face-layer thickness to core thickness in this case of a cylindrical shell.

Spherical Shells

In conformity with customary usage, the following notation is introduced:

$$\left.\begin{array}{lll} \xi_1 = a\phi & \xi_2 = a\theta & \alpha_1 = 1 \\ \alpha_2 = \sin\phi & R_1 = R_2 = a & N_{11} = N_\phi \\ N_{22} = N_\theta & N_{12} = N_{21} = N_{\phi\theta} & Q_1 = Q_\phi \\ Q_2 = Q_\theta & M_{11} = M_\phi & M_{22} = M_\theta \\ M_{12} = M_{\phi\theta} & p_1 = p_\phi & p_2 = p_\theta \\ m_1 = m_\phi & m_2 = m_\theta & u_1 = u \\ u_2 = v & \beta_1 = \beta_\phi & \beta_2 = \beta_\theta \end{array}\right\} \qquad (223)$$

Attention is here restricted to problems with rotational symmetry and the following relations are used:

$$\left.\begin{array}{l} \partial(\ )/\partial\theta = 0 \\ \\ N_{\phi\theta} = Q_\theta = M_{\phi\theta} = p_\theta = m_\theta = v = \beta_\theta = 0 \end{array}\right\} \qquad (224)$$

The differential equations of equilibrium (25) to (28), (31), and (34) become, setting

$$\partial(\ )/\partial\phi = d(\ )/d\phi = (\ )'$$

$$\left(\sin \phi N_\phi\right)' - \cos \phi N_\theta + \sin \phi Q_\phi + a \sin \phi \, p_\phi = 0 \tag{225}$$

$$\left(\sin \phi Q_\phi\right)' - \sin \phi\left(N_\phi + N_\theta\right) + a \sin \phi \, q = 0 \tag{226}$$

$$\left(\sin \phi M_\phi\right)' - \cos \phi M_\theta - a \sin \phi Q_\phi + a \sin \phi \, m_\phi = 0 \tag{227}$$

$$\sigma_{\zeta m} + \left(M_\phi + M_\theta\right)\Big/(h + t)a - s = 0 \tag{228}$$

The stress-strain relations (56), (59) to (61), and (64) to (67) become, if there is set in accordance with equation (63)

$$\lambda_1 = \lambda_2 = \lambda_{12} = \frac{1}{2} \frac{(h + t)t}{a^2} \frac{E_f}{E_c} = \lambda \tag{229}$$

$$\left(1 + \frac{\lambda}{3}\right)N_\phi - \left(\nu - \frac{\lambda}{3}\right)N_\theta = C^*\left(\frac{u' + w}{a} + \frac{h + t}{12a} \frac{q}{E_c}\right) \tag{230}$$

$$\left(1 + \frac{\lambda}{3}\right)N_\theta - \left(\nu - \frac{\lambda}{3}\right)N_\phi = C^*\left(\frac{u \cot \phi + w}{a} + \frac{h + t}{12a} \frac{q}{E_c}\right) \tag{231}$$

$$Q_\phi = (h + t)G_c\left(\beta_\phi + \frac{w' - u}{a}\right) \tag{232}$$

$$(1 + \lambda)M_\phi - (\nu - \lambda)M_\theta = \frac{D^*}{a}\left(\beta_\phi' + \frac{s}{E_c}\right) \tag{233}$$

$$(1 + \lambda)M_\theta - (\nu - \lambda)M_\phi = \frac{D^*}{a}\left(\beta_\phi \cot \phi + \frac{s}{E_c}\right) \tag{234}$$

There is first given a simple special solution of this system of equations and then a generalization is obtained of the two simultaneous equations for Q_ϕ and β_ϕ which are fundamental in the theory of homogeneous isotropic shells.

Uniform stress distribution in a spherical shell.- Set in equations (225) to (234) $p_\phi = m_\phi = 0$ and assume that N_ϕ, N_θ, Q_ϕ, M_ϕ, and M_θ are independent of ϕ. From equation (225) it follows that:

$$\left.\begin{aligned} N_\phi &= N_\theta = N_0 \\ Q_\phi &= 0 \end{aligned}\right\} \tag{235}$$

From equation (226) it follows then that

$$N_0 = \frac{1}{2} aq \tag{236}$$

and from equation (227) it follows that

$$M_\phi = M_\theta = M_0 \tag{237}$$

Equation (228) gives

$$\sigma\zeta_m = s - 2M_0/(h + t)a \tag{238}$$

In equations (230) and (231) set $u = 0$ for reasons of symmetry and obtain

$$\left(1 + \frac{2}{3} \lambda - \nu\right)N_0 = \frac{C^*}{a} w + \frac{t(h + t)}{6a} \frac{E_f}{E_c} q \tag{239}$$

or, with N_0 from equation (236) and λ from equation (229),

$$(C^*/a)w = \frac{1}{2}(1 - \nu)aq \tag{240}$$

Equation (232) is identically satisfied when $\beta_\phi = 0$. Equations (233) and (234), in conjunction with equation (237), give

$$(1 + 2\lambda - \nu)M_0 = (D^*/a)(s/E_c) = \frac{1}{2}\,\frac{t(h + t)^2}{a}\,\frac{E_f}{E_c}\,s$$

or

$$M_0 = \frac{(h + t)a\lambda}{1 + 2\lambda - \nu}\,s \tag{241}$$

Then, from equation (238),

$$\sigma_{\zeta_m} = \frac{(1 - \nu)s}{1 + 2\lambda - \nu} \tag{242}$$

Equation (242) may be compared with equation (136) for the circular ring.

According to equation (46), there are obtained from equations (236) and (241) the following expressions for the stress resultants in the outer ("upper") and inner ("lower") face layers:

$$\left.\begin{aligned} \left(1 + \frac{h + t}{2a}\right)N_u &= a\left(\frac{q}{4} + \frac{\lambda s}{1 + 2\lambda - \nu}\right) \\ \left(1 - \frac{h + t}{2a}\right)N_l &= a\left(\frac{q}{4} - \frac{\lambda s}{1 + 2\lambda - \nu}\right) \end{aligned}\right\} \tag{243}$$

Comparison of these results with the corresponding results for the circular ring (equations (135)) shows that for given values of q and s there is a greater difference between N_u and N_l in the

spherical shell than there is in the circular ring, the reason being the relatively larger influence of the s-term in equation (243).

For a specific example, it is again assumed that the radial load is applied entirely to the inner face so that $q_u = 0$ and, according to equations (20) and (22),

$$\left.\begin{aligned} q &= \left(1 - \frac{h + t}{2a}\right)^2 q_l \\ s &= -\frac{1}{2}\left(1 - \frac{h + t}{2a}\right)^2 q_l \end{aligned}\right\} \tag{244}$$

Substitution of equation (244) in equation (243) gives

$$\left.\begin{aligned} \left(1 + \frac{h + t}{2a}\right)N_u &= \left(1 - \frac{h + t}{2a}\right)^2 \frac{aq_l}{4} \frac{1 - \nu}{1 + 2\lambda - \nu} \\ N_l &= \left(1 - \frac{h + t}{2a}\right)\frac{aq_l}{4} \frac{1 + 4\lambda - \nu}{1 + 2\lambda - \nu} \end{aligned}\right\} \tag{245}$$

As a numerical example, taking $\lambda = 0.0595$, as in the example given in the section entitled "Closed circular ring acted upon by uniform radial load," and $\nu = 1/3$, it is found that the factor in N_l which contains the effect of the core flexibility is $(1 + 0.36)/(1 + 0.18) = 1.15$. Thus, where for the circular ring there was a 6-percent stress increase, there now is a 15-percent stress increase.

Reduction of axisymmetrical problem to two simultaneous equations for Q_ϕ and β_ϕ.- The fundamental results of reference 10 for homogeneous shells may be readily extended to sandwich shells, as follows:

Equations (225) and (226) are used to express N_ϕ and N_θ in terms of Q_ϕ.

$$N_\phi = \cot \phi Q_\phi + F_1(\phi) \tag{246}$$

$$N_\theta = Q_\phi' + F_2(\phi) \tag{247}$$

In equations (246) and (247) the functions F_1 and F_2 are given by

$$F_1 = \frac{a}{\sin^2\phi}\int\left(q\cos\phi - p_\phi \sin\phi\right)\sin\phi\, d\phi \tag{248}$$

$$F_2 = \frac{1}{\cos\phi}\left[\left(\sin\phi\, F_1\right)' + ap_\phi \sin\phi\right] \tag{249}$$

Next the displacement components u and w are expressed in terms of Q_ϕ, by means of equations (230), (231), (246), and (247).

Subtraction of equation (231) from equation (230) gives

$$\begin{aligned}\frac{C^*}{a}(u' - \cot\phi\, u) &= (1 + \nu)\left(N_\phi - N_\theta\right) \\ &= (1 + \nu)\left[-\left(Q_\phi' - \cot\phi Q_\phi\right) + F_1 - F_2\right]\end{aligned} \tag{250}$$

Equation (112) is integrated to

$$(C^*/a)u = -(1 + \nu)\left(Q_\phi + F_3\right) \tag{251}$$

where F_3 is given by

$$F_3 = -\sin\phi\int\frac{F_1(\phi) - F_2(\phi)}{\sin\phi}\, d\phi \tag{252}$$

Equations (251) and (252) are introduced into equation (231) and the following expression is obtained for w:

$$(C^*/a)w = (1 + \lambda/3)\left(\cot\phi Q_\phi + Q_\phi'\right) + F_4 \tag{253}$$

where F_4 is given by

$$F_4 = -\frac{(h + t)}{12}\frac{q}{E_c} + (1 + \nu)\cot\phi\, F_3 + \left(1 + \frac{\lambda}{3}\right)F_2 - \left(\nu - \frac{\lambda}{3}\right)F_1 \tag{254}$$

Equations (251) and (253) are introduced into equation (232) for Q_ϕ and the first of the two simultaneous equations for Q_ϕ and β_ϕ is obtained in the form

$$\frac{Q_\phi}{(h + t)G_c} = \beta_\phi + \frac{1}{C^*}\left[\left(1 + \frac{\lambda}{3}\right)\left(\cot\phi\, Q_\phi + Q_\phi'\right)' + F_4' + (1 + \nu)\left(Q_\phi + F_3\right)\right]$$

which may be rearranged to read[17]

$$Q_\phi'' + \cot\phi\, Q_\phi' - \left[\cot^2\phi - \frac{\nu - \lambda/3}{1 + \lambda/3} + \frac{C^*}{G_c}\frac{1}{(h + t)(1 + \lambda/3)}\right]Q_\phi + \frac{C^*}{1 + \lambda/3}\beta_\phi = F_5(\phi) \tag{255}$$

The function F_5 is given by

$$F_5 = \frac{(1 + \nu)F_3 + F_4}{1 + \lambda/3} \tag{256}$$

Introducing the operator

$$L \equiv (\ )'' + \cot\phi(\ )' - \cot^2\phi(\ )$$

[17]When $\lambda = 0$ and $G_c = \infty$ and when no external loads are present, this equation checks with the first of equations (g) on page 469 of reference 9.

equation (255) may finally also be written

$$(L - \mu_1)Q_\phi + \frac{C^*}{1 + \lambda/3}\beta_\phi = F_5(\phi) \tag{257}^*$$

where

$$\mu_1 = \left(\frac{2t}{h + t}\frac{E_f}{G_c} - \nu + \frac{\lambda}{3}\right)\frac{1}{1 + \lambda/3} \tag{258}$$

The second of the two simultaneous equations is obtained somewhat more directly as follows: Write equations (233) and (234) in the form

$$M_\phi = \frac{D^*/a}{1 - \nu^2 + 2\lambda(1 + \nu)}\Big[(1 + \lambda)\beta_\phi' + (\nu - \lambda)\cot\phi\,\beta_\phi$$
$$+ (1 + \nu)s/E_c\Big] \tag{259}$$

$$M_\theta = \frac{D^*/a}{1 - \nu^2 + 2\lambda(1 + \nu)}\Big[(1 + \lambda)\cot\phi\,\beta_\phi + (\nu - \lambda)\beta_\phi'$$
$$+ (1 + \nu)s/E_c\Big] \tag{260}$$

Introduce equations (259) and (260) into the moment equilibrium equation (228) and obtain

$$\frac{D^*/a}{1 - \nu^2 + 2\lambda(1 + \nu)}\Big[\beta_\phi'' + \cot\phi\,\beta_\phi' - \left(\cot^2\phi + \frac{\nu - \lambda}{1 + \lambda}\right)\beta_\phi + (1 + \nu)s'/E_c$$
$$+ (1 + \nu)s'/E_c\Big] - aQ_\phi + am_\phi = 0 \tag{261}$$

Again, using the operator L, this may be written in the form

$$\left(L - \frac{\nu - \lambda}{1 + \lambda}\right)\beta_\phi - \frac{a^2}{D^*}\left[1 - \nu^2 + 2\lambda(1 + \nu)\right]Q_\phi = F_6(\phi) \qquad (262)^*$$

The function F_6 is given by

$$F_6 = -(1 + \nu)\frac{s'}{E_c} - \frac{a^2}{D^*}\left[1 - \nu^2 + 2\lambda(1 + \nu)\right]m_\phi \qquad (263)$$

Equation (262) may be compared with the second of equations (g) on page 469 of reference 9.

Analysis of edge effect for spherical shell.– The special case of no distributed surface load and no concentrated load at the apex of the shell is obtained by setting

$$F_5 = F_6 = 0$$

Following again a known procedure from the theory without transverse stress deformation, there may be set

$$\left.\begin{aligned} Q_\phi &= \frac{Q_1}{\sqrt{\sin \phi}} \\ \beta_\phi &= \frac{\beta_1}{\sqrt{\sin \phi}} \end{aligned}\right\} \qquad (264)$$

$$\left.\begin{aligned} Q_\phi' &= \frac{Q_1'}{\sqrt{\sin \phi}} - \frac{1}{2}\cot \phi \frac{Q_1}{\sqrt{\sin \phi}} \\ Q_\phi'' &= \frac{Q_1''}{\sqrt{\sin \phi}} - \cot \phi \frac{Q_1'}{\sqrt{\sin \phi}} + \left(\frac{3}{4}\cot^2\phi + \frac{1}{2}\right)\frac{Q_1}{\sqrt{\sin \phi}} \end{aligned}\right\} \qquad (265)$$

with corresponding formulas for $\beta_\phi{}'$ and $\beta_\phi{}''$. Introduction of equation (264) into equations (257) and (262) gives

$$Q_1{}'' + \left(\mu_1 + \frac{1}{2} - \frac{3}{4}\cot^2\phi\right)Q_1 + \frac{C^*}{1 + \lambda/3}\beta_1 = 0 \tag{266}$$

$$\beta_1{}'' - \left(\frac{\nu - \lambda}{1 + \lambda} - \frac{1}{2} + \frac{3}{4}\cot^2\phi\right)\beta_1 - \frac{a^2}{D^*}\left[1 - \nu^2 + 2\lambda(1 + \nu)\right]Q_1 = 0 \tag{267}$$

Attention is restricted to the cases for which $F_5 = F_6 = 0$ in equations (266) and (267) and the problem is considered of the shell subject to edge loads $\left(M_\phi\right)_0$, $\left(Q_\phi\right)_0$, and $\left(N_\phi\right)_0$ at a section $\phi = \phi_0$. Assuming that $\cot\phi_0$ is not large compared with unity[18] and that the effect of the edge loads is restricted to a narrow edge zone so that $\left|Q_1\right| \ll \left|Q_1{}''\right|$, $\left|\beta_1\right| \ll \left|\beta_1{}''\right|$, equations (266) and (267) may be simplified to

$$Q_1{}'' - \tilde{\mu}_1 Q_1 + \frac{C^*}{1 + \lambda/3}\beta_1 = 0 \tag{268)*}$$

$$\beta_1{}'' - \frac{a^2}{D^*}\left[1 - \nu^2 + 2\lambda(1 + \nu)\right]Q_1 = 0 \tag{269)*}$$

Equations (268) and (269) show that the influence of finite $E_c (\lambda \neq 0)$ in the edge-effect problem consists, except in extreme circumstances, in minor modifications of the results for $E_c = \infty$. The quantity $\tilde{\mu}_1$ which represents the influence of finite G_c and which is

$$\tilde{\mu}_1 = \frac{2t}{h + t}\frac{E_f}{G_c} \tag{270}$$

may, however, in practical cases be large compared with unity and not of negligible influence on the results.

[18]When $\cot\phi_1 \gg 1$ the shell is termed a "shallow" shell which is not considered in what follows.

Equations (268) and (269) may be compared with equations (180) and (182) for the cylindrical shell. This comparison shows that the influence of finite G_c in the edge-effect problem is of the same nature for the spherical and cylindrical shells. Thus, results of the same quantitative nature will be obtainable as in the section on cylindrical shells under the headings entitled "Finite circular cylindrical shell acted upon by edge moments and forces" and "Semi-infinite shell acted upon by edge bending moment and shear force."

This work is not herein carried further to specific applications. It is apparent that such applications may be worked out with hardly any more difficulty than when the effect of the core deformability is not taken into account.

CONCLUDING REMARKS

A system of basic equations has been derived for the analysis of small-deflection problems of sandwich-type thin shells. This system of equations reduces to Love's theory of thin shells when the transverse shear and normal stress deformability of the core of the sandwich is of negligible importance. The system of basic equations has been applied to a number of specific problems from the theory of plates, circular rings, circular cylindrical shells, and spherical shells, and it has been found that the effects of both transverse shear and transverse normal stress deformation may be of such magnitude that an analysis which disregards them gives values for deflections and stresses which are appreciably in error.

Numerical calculations have been in the nature of sample calculations, illustrating both the use of the equations and the possible effects of using them. Examples have been chosen from the point of view of relative simplicity as well as with the thought to illustrate most clearly the consequences of the extra deformations which have been taken into account. It is unavoidable that, in so doing, some of the examples may be of little interest for aircraft structural analysis and that some problems may not have been analyzed which would have well fitted within the contents of this report and which at the same time would have been of considerable practical importance.

The general analysis has been restricted by the following two order-of-magnitude relations: (1) $t/h \ll 1$ and (2) $tE_f/hE_c \gg 1$, where t is the face-layer thickness, h is the core-layer thickness, E_f is the elastic modulus of the isotropic face-layer material, and E_c is the elastic modulus in the transverse direction of the core-layer material. Therewith it is felt that very likely nearly all situations have been covered in which the effect of transverse core flexibility is of significant practical importance. It is evident, however, that if desired the theory could be extended so as

to include cases where one or both of these two order-of-magnitude relations are not satisfied. The main limitation of the present analysis is the omission of all finite-deflection and instability effects.

Massachusetts Institute of Technology
Cambridge, Mass., May 26, 1947

REFERENCES

1. Reissner, Eric: On Bending of Elastic Plates. Quarterly Appl. Math., vol. V, no. 1, April 1947, pp. 55–68.

2. Reissner, Eric: A New Derivation of the Equations for the Deformation of Elastic Shells. Am. Jour. Math., vol. 63, no. 1, Jan. 1941, pp. 177–184.

3. Love, A. E. H.: A Treatise on the Mathematical Theory of Elasticity. Fourth ed., The Univ. Press (Cambridge), 1934, p. 90.

4. Trefftz, E.: Ableitung der Schalenbiegungs–gleichungen mit dem Castigliano' schem Prinzip. Z.f.a.M.M., Bd. 15, 1935, pp. 101–108.

5. Donnell, L. H.: Discussion of "The Effect of Transverse Shear Deformation on the Bending of Elastic Plates" by Eric Reissner. Jour. Appl. Mech., vol. 13, no. 3, Sept. 1946, pp. A–249 – A–250.

6. Beskin, Leon: General Circular Ring Analysis. Aircraft Eng., vol. XVII, no. 194, April 1945, pp. 126–132.

7. Reissner, H.: Formänderung einer dünnwandigen, an den Rändern frei aufliegenden, beliebig belasteten Zylinderschale. Z.f.a.M.M., Bd. 13, 1933, pp. 133–138.

8. Biezeno, C. B., and Koch, J. J.: Some Explicit Formulas, of Use in the Calculation of Arbitrarily Loaded Thin–Walled Cylinders. Nederl. Akad. Wetensch. Proc., vol. 44, 1941, pp. 505–512.

9. Timoshenko, S.: Theory of Plates and Shells. McGraw–Hill Book Co., Inc., 1940.

10. Reissner, H.: Spannungen in Kugelschalen (Kuppeln). Müller–Breslau–Festschrift, 1912, pp. 181–193.

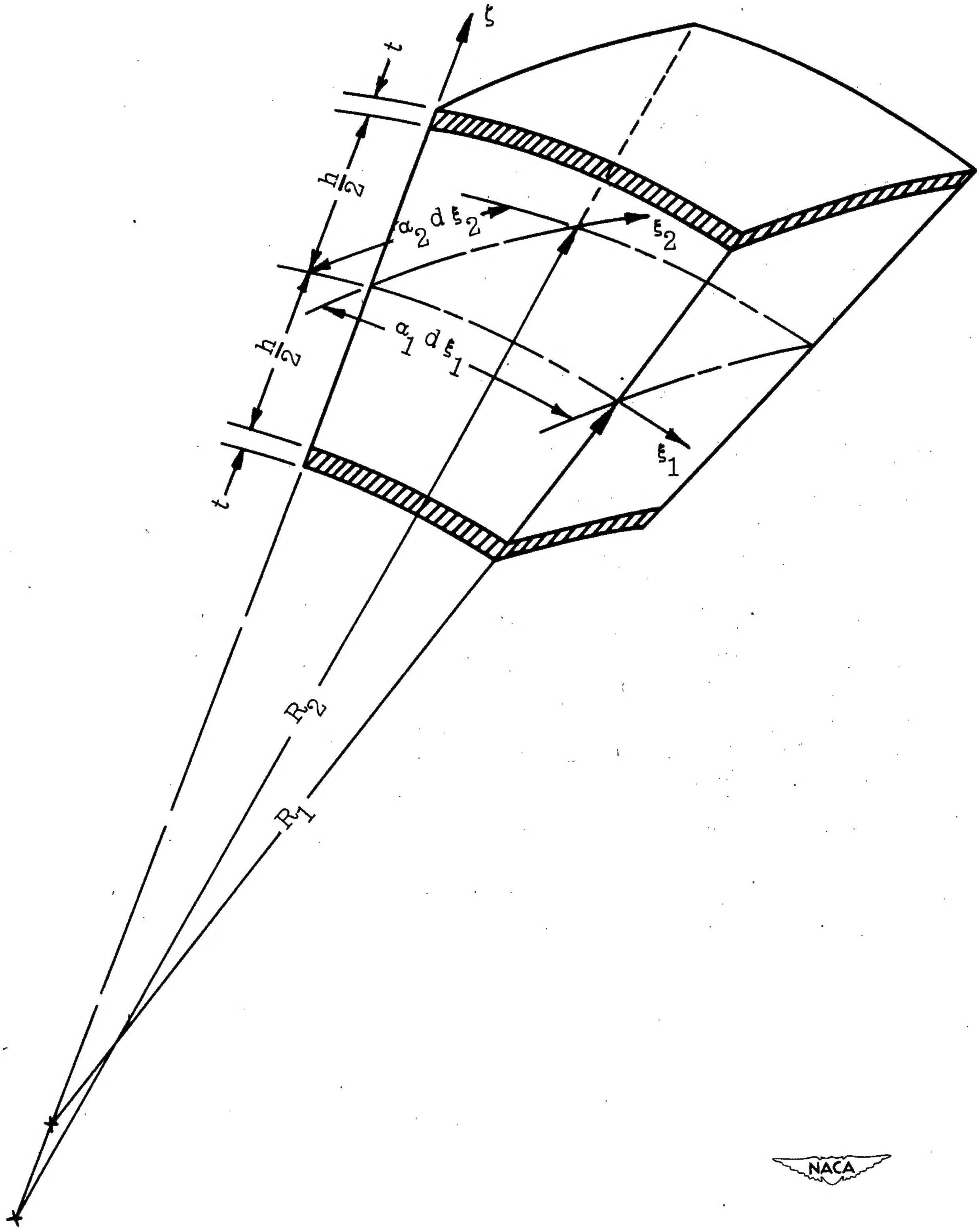

Figure 1.- Element of composite shell, showing coordinates and dimensions.

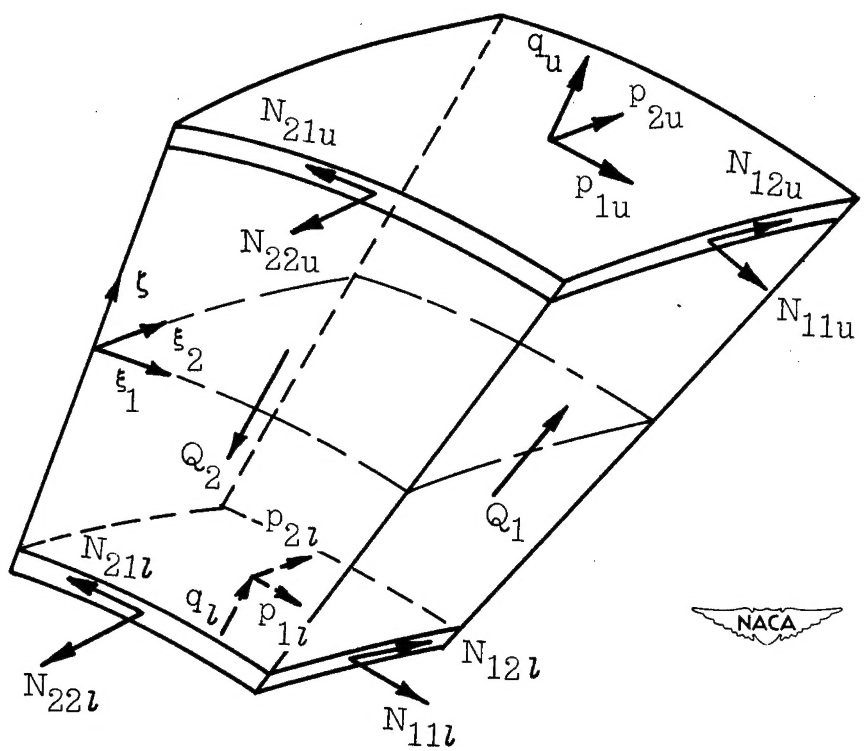

Figure 2.- Element of composite shell, showing location and orientation of stress resultants in face layers and core layer and orientation of external loads.

Abstract

A theory is developed for small bending and stretching of sandwich-type shells. This theory is an extension of the known theory of homogeneous thin elastic shells. It is found that two effects are important in the present problem, which have not been considered previously in the theory of curved shells: (1) The effect of transverse shear deformation and (2) the effect of transverse normal stress deformation.

The general results are applied to the solution of problems concerning flat plates, circular rings, circular cylindrical shells, and spherical shells.

Abstract

A theory is developed for small bending and stretching of sandwich-type shells. This theory is an extension of the known theory of homogeneous thin elastic shells. It is found that two effects are important in the present problem, which have not been considered previously in the theory of curved shells: (1) The effect of transverse shear deformation and (2) the effect of transverse normal stress deformation.

The general results are applied to the solution of problems concerning flat plates, circular rings, circular cylindrical shells, and spherical shells.